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Abstract

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ON SINGULAR EQUATIONS WITH A SYMBOL THAT VANISHES

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By now, multidimensional integral singular equations with a symbol nowhere equal to zero have been well studied (see (1)). In this note singular equations in the plane are considered under the condition that the symbol may vanish on a finite number of rays. The equations are studied in spaces of generalized functions. The note is devoted mainly to equations of convolution type. In contrast to the case when the symbol does not degenerate, the solution of such equations is not determined uniquely. The general solution contains functions of the type of a plane wave. It is therefore natural to append to the equation additional conditions that would ensure its unique solvability. For equations of convolution type, three well-posed problems are considered. In conclusion, one problem is presented for an equation with a symbol weakly depending on the pole.

1. Let polar coordinates (ρ, θ) be introduced in the plane E_2. Denote by $\mathcal{L}_2^{(l)}$ (l a positive integer) the space of functions on the circle that are square-summable together with derivatives up to order l inclusive. The space $\mathcal{L}_2^{(l)}$, endowed with the norm

$$|\varphi|_l^2 = \int_0^{2\pi} \sum_{k=0}^{l} \left| \frac{\partial^k \varphi}{\partial \theta^k} \right|^2 d\theta,$$

is a normed ring with respect to ordinary multiplication. Let $\mathcal{L}_2^{(-l)}$ be the space conjugate to $\mathcal{L}_2^{(l)}$ with respect to the scalar product $[f, \varphi]$ in $\mathcal{L}_2$ on the circle.

We shall consider functions defined on E_2 as abstract functions of one variable ρ, whose values belong to various spaces of functions on the circle. Introduce the space $H(\mathcal{L}_2^{(l)})$ of functions of the variable ρ with values in $\mathcal{L}_2^{(l)}$, square-summable on the half-axis $(0, \infty)$ with weight ρ. This space is provided with the norm

$$\|u\|_l^2 = \int_0^{\infty} |u|_l^2 \, \rho \, d\rho.$$

The space of functionals $H(\mathcal{L}_2^{(-l)})$ is defined analogously.

The Fourier transform $\mathcal{F}u$ is defined in the space $H(\mathcal{L}_2^{(l)})$ and maps it isometrically onto itself. The space $H(\mathcal{L}_2^{(-l)})$ is conjugate to $H(\mathcal{L}_2^{(l)})$ with respect to the scalar product

$$(v, u) = \int_0^\infty [v, u]\rho \, d\rho.$$

Therefore the Fourier transform is also defined in $H(\mathcal{L}_2^{(-l)})$. In order that a homogeneous function of degree zero $\Phi(z_1, z_2) = \Phi(\theta)$ (θ is the polar angle of the point z) be a multiplier in $H(\mathcal{L}_2^{(l)})$, it is necessary and

it is sufficient that $\Phi(\theta)$ belong to $\mathcal{L}_2^{(l)}$. Consequently, in the spaces $H(\mathcal{L}_2^{(l)})$ and $H(\mathcal{L}_2^{(-l)})$ there is defined and bounded the singular integral operator $A = \mathcal{F}^{-1}\Phi\mathcal{F}$ with symbol $\Phi(\theta) \in \mathcal{L}_2^{(l)}$.

2. Suppose that the symbol $\Phi(\theta)$ vanishes on the rays $\theta = \chi_1, \chi_2, \dots, \chi_N$; denote by l_j the multiplicity of the zero $\theta = \chi_j$, and let

$$l = \max_j \{l_j\}.$$

If $f \in L_2(E_2)$, then there exists a solution of the equation $Au = f$ belonging to the space $H(\mathcal{L}_2^{(-l)})$. The general solution of the homogeneous equation has the form

$$\mathcal{F}u = \sum_{j=1}^{N} \sum_{k=0}^{l_j-1} c_{kj}(\rho)\, \delta^{(k)}(\theta - \chi_j), \tag{1}$$

where $c_{kj}(\rho)$ are arbitrary functions from $L_2[(0, \infty); \rho]$.

Assume that for all λ the function $\Phi(\theta) - \lambda$ vanishes only on a finite number of rays, and that the multiplicities of the zeros do not exceed a fixed number l. Then the spectrum of the operator A in the space $H(\mathcal{L}_2^{(-l)})$ coincides with the set of values of the symbol $\Phi(\theta)$, and every point of the spectrum is an eigenvalue of infinite multiplicity. The form of the eigenfunctions follows directly from (1).

3. We pose the following problem:

$$Au = f, \qquad [A_k u, 1] = \psi_k(\rho) \quad \left(k = 1, \dots, s = \sum_{j=1}^{N} l_j\right), \tag{2}$$

where A_k are singular operators with symbols $\Phi_k(\theta) \in \mathcal{L}_2^{(l)}$, and $\psi_k(\rho)$ are known functions from $L_2[(0, \infty); \rho]$.

Let B be a square matrix of order s of the form $B = (B_1, \dots, B_N)$, where

$$B_j = \begin{pmatrix} \Phi_1(\chi_j) \dots \Phi_1^{(l_j-1)}(\chi_j) \\ \Phi_2(\chi_j) \dots \Phi_2^{(l_j-1)}(\chi_j) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \Phi_s(\chi_j) \dots \Phi_s^{(l_j-1)}(\chi_j) \end{pmatrix}.$$

For the existence and uniqueness of a solution of problem (2) for arbitrary $\psi_k(\rho)$, it is necessary and sufficient that the determinant $\det B$ be nonzero. If this condition is fulfilled, the solution depends continuously on f and ψ_k.

Considerations analogous to those carried out in items 2, 3 are also possible in the space of generalized functions over a countably normed space in which the topology is given by a countable system of norms $\|u\|_l$ $(l = 0, 1, \dots)$.

4. Suppose that the multiplicity of each zero $\theta = \chi_j$ $(j = 1, \dots, N)$ of the symbol Φ is equal to one. Choose Cartesian coordinates so that none of the rays $\theta = \chi_j$ lies on the axis $x_1 = 0$, and let the rays $\theta = \chi_j$ $(j = 1, \dots, M)$ project onto the positive half-axis $x_2 = 0$, while the remaining rays project onto the negative one.

In posing the second and third problems we shall use the fact that $(-\Delta + E)^{-2} u \in \mathcal{L}_2^{(1)}$ for almost all ρ for any $u \in H(\mathcal{L}_2^{(-1)})$, and therefore the function $(-\Delta + E)^{-2} u$ is defined almost everywhere on every straight line.

We shall seek a solution of the equation

$$Au = 0, \tag{3}$$

satisfying the conditions

$$(-\Delta + E)^{-2} A_k \operatorname{Re} u\big|_{x=0} = \varphi_k(x_1) \quad (k = 1, \dots, N), \tag{4}$$

where A_k are real singular operators with symbols $\Phi_k(\theta)$. By $\varphi_k(x_1)$ are denoted prescribed real functions from $L_2(-\infty, +\infty)$,

subject to the condition

$$\int_{-\infty}^{+\infty} |\widetilde{\varphi}_k|^2 (1 + t^2)^2 \frac{dt}{|t|} < \infty, \tag{5}$$

where $\widetilde{\varphi}_k$ is the Fourier transform of the function φ_k.

For the existence and uniqueness of the solution of problem (3)–(4) it is necessary and sufficient that the determinant

$$\det \begin{pmatrix} \Phi_1(\theta_1) & \dots & \Phi_1(\theta_M) & \Phi_1(\theta_{M+1} + \pi) & \dots & \Phi_1(\theta_N + \pi) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \Phi_N(\theta_1) & \dots & \Phi_N(\theta_M) & \Phi_N(\theta_{M+1} + \pi) & \dots & \Phi_N(\theta_N + \pi) \end{pmatrix}$$

not be equal to zero. Under this condition the solution depends continuously on φ_k.

If the number of zeros of the symbol is even and the multiplicity of each zero is equal to one, then one can pose one more correct problem. Choose Cartesian coordinates so that M is equal to $N/2$ (this is always possible). We shall seek a solution of the equation $Au = 0$ such that

$$(-\Delta + E)^{-2} A_k u\big|_{x_2=0} = \varphi_k(x_1) \qquad (k = 1), \dots, N/2), \tag{6}$$

where A_k are singular operators (not necessarily real) with symbols Φ_k. By $\varphi_k(x_1)$ are denoted prescribed complex functions satisfying condition (5).

For the existence and uniqueness of the solution of problem (3)–(6) for all φ_k, it is necessary and sufficient that the determinants

$$\det \|\Phi_k(\chi_j)\|, \qquad \det \|\Phi_k(\chi_{N/2+j})\| \qquad (k, j = 1, \dots, N/2)$$

not be equal to zero.

5. Let $f(x, z)$ and $\Phi(x, z)$ be functions homogeneous of degree zero with respect to z. Consider the singular operator

$$Au = a(x)u(x) + \int_{E_2} \frac{f(x, y - x)}{r^2} u(y)\, dy$$

with an infinitely differentiable symbol $\Phi(x, z) \equiv \Phi(x, \theta)$. Suppose that, for sufficiently large $|x|$, the symbol depends only on z. The operator A is bounded in the spaces $H(\mathcal{L}_2^{(1)})$ and $H(\mathcal{L}_2^{(-1)})$.

A bounded operator T, acting from $H(\mathcal{L}_2^{(-1)})$ into $L_2(E_2)$, will be called smoothing; the symbol of T, by definition, is equal to zero. A general singular operator will mean an operator of the form $A + T$. For such operators the rule of multiplication of symbols is valid.

The curve $F(x) = \text{const}$ will be called characteristic for the operator $A + T$ if

$$\Phi(x, \operatorname{grad} F) = 0. \tag{7}$$

Equation (7) remains invariant under a change of coordinates. Suppose that the symbol $\Phi(x, \theta)$ of the operator A has the form

$$\Phi(x, \theta) = (e^{i\chi(x)} - e^{i\theta})\Psi(x, \theta) \qquad (\operatorname{Im} \chi \equiv 0),$$

where $|\Psi(x, \theta)| > 0$, and that the characteristic curves are given by the equation $x_1 \cos \omega(x) + x_2 \sin \omega(x) = \text{const}$. The functions $\Psi(x, \theta) - \Psi(\infty, \theta)$ and

$\omega(x)-\omega(\infty)$ are identically equal to zero for large $|x|$, and, together with their derivatives with respect to x and θ of sufficiently high order, do not exceed in absolute value a prescribed small positive number.

Under these assumptions, the problem

$$Au = f, \qquad [u,1] = \psi(\rho),$$

where $f \in L_2(E_2)$, $\psi(\rho) \in L_2[(0,\infty);\rho]$, is uniquely solvable in the space $H(\mathcal{L}_2^{(-1)})$.

For the proof, by means of a special change of variables preserving the space $H(\mathcal{L}_2^{(-1)})$, we straighten the characteristic curves of the operator A. Then, multiplying A by a certain operator with a nondegenerate symbol, we obtain a singular operator whose symbol does not depend on the pole. Finally, using the results of §§ 2, 3 and the method of successive approximations, we prove the existence and uniqueness of the solution of the problem.

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References

1. S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.