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I. Ts. Gohberg, M. G. Kreĭn

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Abstract

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MATHEMATICS

I. Ts. Gohberg, M. G. Kreĭn

ON THE MULTIPLICATIVE REPRESENTATION OF CHARACTERISTIC FUNCTIONS OF OPERATORS CLOSE TO UNITARY ONES

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In the present communication it will be shown that the authors' investigations ([1]) on the factorization of operators, in combination with various investigations of other authors ([2]-[4]), make it possible to obtain a multiplicative representation of the characteristic functions of operators of a comparatively broad class.

1. Let $\mathfrak{S}$ be an arbitrary **symmetrically normed ideal** (see ([1])) of the ring $\mathfrak{R}$ of all linear bounded operators acting in the separable Hilbert space $\mathfrak{H}$. As is known, $\mathfrak{S} \subseteq \mathfrak{S}_\infty$, where $\mathfrak{S}_\infty$ is the ideal of all completely continuous operators of the ring.

By $\mathfrak{U}(\mathfrak{S})$ we shall denote the class of all operators $T \in \mathfrak{R}$ possessing the following properties: 1) $T^*T - I \in \mathfrak{S}$; 2) in the disk $|\lambda| < 1$ there is at least one regular point of the operator T. The following assertions are easily proved.

If an operator $T \in \mathfrak{U}(\mathfrak{S})$, then in this and only in this case it admits a polar representation of the form $T = U(I + K)$, where U is a unitary operator and $K \in \mathfrak{S}$. Therefore, if $T \in \mathfrak{U}(\mathfrak{S})$, then also $T^* \in \mathfrak{U}(\mathfrak{S})$, and consequently $TT^* - I \in \mathfrak{S}$.

If $T \in \mathfrak{U}(\mathfrak{S})$, then the set of points of its spectrum $\sigma(T)$ not lying on the unit circle is either empty or consists of isolated eigenvalues to which finite-dimensional normally splitting root subspaces correspond.

An operator whose entire spectrum lies on the unit circle will be called an operator with **unitary spectrum**.

Let the monotone chain $\{P_t\}_{\alpha \le t \le 2\pi+\alpha}$ ($P_\alpha = 0, P_{2\pi+\alpha} = I$; $P_{t_1} \le P_{t_2}$ for $t_1 < t_2$) of orthogonal projectors be an eigenchain (see ([1])) of some operator $T(\in \mathfrak{R})$ with unitary spectrum: $TP_t = P_{tTP}t$. We agree to say that the chain $\{P_t\}_{\alpha \le t \le 2\pi+\alpha}$ **separates** the spectrum $\sigma(T)$ of the operator T, if the spectrum of the restriction of T to the subspace $P_t\mathfrak{H}$ lies on the arc $e^{i\tau}$ $(\alpha \leq \tau \leq t)$, while the spectrum of the restriction of the operator $(I-P_t)T(I-P_t)$ to the subspace $(I-P_t)\mathfrak{H}$ lies on the complementary arc $e^{i\tau}$ $(t \leq \tau \leq 2\pi + \alpha)$.

Operators possessing eigenchains separating their spectrum form a sufficiently broad class. The method of V. I. Macaev ([2]) for estimating the resolvents of linear operators and the theorem of Yu. I. Lyubich and V. I. Macaev ([3]) make it possible, in particular, to assert that every operator $T \in \mathfrak{U}(\mathfrak{S}_\omega)$ (for the definition of $\mathfrak{S}_\omega$, see ([1])) with unitary spectrum, for every α $(0 \leq \alpha < 2\pi)$, possesses an eigenchain separating the spectrum.*

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* Moreover, the operator T is an S-operator (in the terminology of Yu. I. Lyubich and V. I. Macaev ([3])).

Theorem 1. In order that an operator $T \in \mathfrak{U}(\mathfrak{S}_\infty)$ be representable in the form $T = U(I+V)$, where V is a Volterra operator with maximal chain of eigenspaces $\mathfrak{P}$, and U is the unitary operator defined by the equality

$$U = \int_{\mathfrak{P}} \exp(i\varphi(P))\, dP, \tag{1}$$

in which $\varphi(P)$ $(P \in \mathfrak{P};\ 0 \leq \varphi(P) \leq 2\pi)$ is a nondecreasing continuous-from-the-left function, it is necessary and sufficient that the spectrum of the operator T be unitary and that the operator T possess a chain $\{P_t\}_{0\leq t\leq 2\pi}$ separating its spectrum.

An analogous theorem for bounded operators with real spectrum and completely continuous imaginary component was proved earlier by M. S. Brodskii ([5]).

If the operator T admits the representation indicated in Theorem 1, then as the chain $\{P_t\}_{0\leq t\leq 2\pi}$ separating its spectrum one may choose a certain part of the chain $\mathfrak{P}$. Conversely, if a certain chain $\{P_t\}_{0\leq t\leq 2\pi}$ separating the spectrum of the operator T is given, then every maximal eigenspace chain $\mathfrak{P}$ of the operator T containing $\{P_t\}_{0\leq t\leq 2\pi}$ generates the required representation of the operator T.

Let us also note that in the representation $A = U(I+V)$, which is discussed in Theorem 1, the operator $I+V$ can be obtained from the formula

$$(I+V)^{-1} = I + \int_{\mathfrak{P}} (I - PHP)^{-1} PH\, dP, \tag{2}$$

where $H = I - T^*T$. The last formula follows from general propositions on factorization from ([1]). Theorem 1 itself, with the addition of (2), was essentially indicated in the same paper of the authors ([1]).

2. Starting from Theorem 1 and formula (2), one can obtain a multiplicative representation of the characteristic function of an operator $T \in \mathfrak{U}(\mathfrak{S})$ with unitary spectrum.

For the purpose of simplifying the exposition we shall restrict ourselves to the consideration of such functions only for contractions T ($|T| \leq 1$). For a contraction T the characteristic function $\theta_T(\lambda)$ may be defined as a function of λ ($|\lambda| < 1$), whose values are again contractions acting from the closure $\mathfrak{K}_T$ of the range of the operator $I - T^*T$ ($\mathfrak{K}_T = (I - T^*T)\mathfrak{H}$) into the closure $\mathfrak{K}_{T^*} = (I - TT^*)\mathfrak{H}$ according to the following formula (see ([4,6–12]))*:

$$\theta_T(\lambda) = \left[-T + \lambda(I - TT^*)^{1/2}(I - \lambda T^*)^{-1}(I - T^*T)^{1/2}\right] \mid \mathfrak{K}_T. \tag{3}$$

For arbitrary μ and λ ($|\mu|, |\lambda| < 1$) the identity

$$\theta_T^*(\mu)\theta_T(\lambda) = \left[I - (I - T^*T)^{1/2}(I - \bar{\mu}T)^{-1}(I - \lambda T^*)^{-1}(I - T^*T)^{1/2}\right] \mid \mathfrak{K}_T, \tag{4}$$

holds, from which, in particular (for $\lambda = \mu$), it follows that $|\theta_T(\lambda)| \leq 1$, and moreover that $|\theta_T(\lambda)f| < |f|$ for $f \neq 0$ ($f \in \mathfrak{K}_T$).

A contraction T is called **simple** (in the terminology of ([4]), **completely non-unitary**) if it is not unitary on any of its invariant subspaces. For every contraction T, $\theta_T(\lambda) = \theta_{T_0}(\lambda)$, where T_0 is the maximal simple part of the operator T. A simple contraction is determined by its characteristic function $\theta_T(\lambda)$ up to unitary equivalence (see Theorem 3 in ([4]); a more general theorem is proved in ([9])).

Let $\mathfrak{K}$ and $\mathfrak{K}_*$ be certain unitary spaces (either finite-dimensional or separable Hilbert spaces), and let $\theta(\lambda)$ ($|\lambda| < 1$) be a certain function whose values are linear bounded–

* By $A|\mathfrak{K}$ we denote the restriction of an operator A, acting in $\mathfrak{H}$, to the subspace $\mathfrak{K}$.

operators acting from $\mathfrak{K}$ into $\mathfrak{K}_*$. The following fundamental theorem, proved in ([4]) (Theorem 2), holds:

A. In order that there correspond to the spaces $\mathfrak{K}$, $\mathfrak{K}_*$ and the function $\theta(\lambda)$ a simple contraction T such that

$$\theta(\lambda)U = \theta_T(\lambda)U_* \qquad (|\lambda| < 1), \tag{5}$$

where U and U_* are certain operators mapping unitarily $\mathfrak{K}$ and $\mathfrak{K}_*$, respectively, onto $\mathfrak{K}_T$ and $\mathfrak{K}_{T^*}$, it is necessary and sufficient that the function $\theta(\lambda)$ have the following properties: 1) the function $\theta(\lambda)$ is holomorphic inside the unit disk; 2) $|\theta(\lambda)| \leq 1$ ($|\lambda| = 1$); 3) $|\theta(0)f| < |f|$ for $f \neq 0$ ($f \in \mathfrak{K}$).

Denote by $\mathcal{C}(\mathfrak{S})$ the totality of all contractions $T \in \mathfrak{A}(\mathfrak{S})$. Thus, $T \in \mathcal{C}(\mathfrak{S})$ if: 1) $|T| \leq 1$; 2) $I - T^*T \in \mathfrak{S}$; 3) the operator T has at least one regular point inside the disk $|\lambda| < 1$.

Let A be a linear operator acting from $\mathfrak{K}$ into $\mathfrak{K}_*$. We shall agree to write $A \in \mathcal{C}(\mathfrak{S}; \mathfrak{K}, \mathfrak{K}_*)$ if either $\dim \mathfrak{K} = \dim \mathfrak{K}_* < \infty$, or $W_* A W^{-1} \in \mathcal{C}(\mathfrak{S})$, where W_* and W are any operators mapping unitarily $\mathfrak{K}_*$ and $\mathfrak{K}$ onto $\mathfrak{H}$. It is easy to see that if, for at least one such pair of unitary operators W_* and W, the condition $W_* A W^{-1} \in \mathcal{C}(\mathfrak{S})$ is fulfilled, then it is fulfilled for every other such pair.

Proposition A may be supplemented by the following:

B. In order that an operator-valued function $\theta(\lambda)$, acting from $\mathfrak{K}$ into $\mathfrak{K}_*$, correspond to a simple contraction $T \in \mathcal{C}(\mathfrak{S})$, for which (5) holds, it is necessary and sufficient that, in addition to properties 1), 2), and 3), it also have the property: 4) there exists a point λ_0 such that $\theta(\lambda_0)$ maps $\mathfrak{K}$ one-to-one onto $\mathfrak{K}_*$ and $\theta(\lambda_0) \in \mathcal{C}(\mathfrak{S}; \mathfrak{K}, \mathfrak{K}_*)$. If properties 1)–4) are fulfilled, then $\theta(\lambda) \in \mathcal{C}(\mathfrak{S}; \mathfrak{K}, \mathfrak{K}_*)$ for all λ ($|\lambda| < 1$).

Proposition B can be given a proof independent of Theorem A, which leads to the construction of an operator T different from that given in ([4]) and related in its idea to the constructions proposed in ([11, 12]) for the recovery of a bounded operator with a completely nonunitary imaginary component from its characteristic function.

Let us present this construction. Since $\theta(0) \in \mathcal{C}(\mathfrak{S}; \mathfrak{K}, \mathfrak{K}_*)$, there is a unitary mapping E of the space $\mathfrak{K}_*$ onto $\mathfrak{K}$ such that $E\theta(0)$ will be a nonnegative operator with spectrum lying in the interval $(0, 1)$, and in addition $I - E\theta(0) \in \mathcal{C}(\mathfrak{S}; \mathfrak{K}, \mathfrak{K})$. If we form the function

$$F(\lambda) = [I - E\theta(\lambda)][I + E\theta(\lambda)]^{-1},$$

then it will be holomorphic inside the unit disk and take there values in $\mathcal{C}(\mathfrak{S}; \mathfrak{K}, \mathfrak{K})$, and, moreover, have the property that $F(\lambda) + F^*(\lambda) \geq 0$. It can then be proved that there exist a unitary operator U, acting in the Hilbert space $\mathfrak{H}$, and a contraction R, acting from $\mathfrak{K}$ into $\mathfrak{H}$, such that

$$F(\lambda) = R^*(U + \lambda I)(U - \lambda I)^{-1} R.$$

It turns out that the contraction

$$T = U(T - RR^*)(I + RR^*)^{-1} \in \mathcal{C}(\mathfrak{S})$$

will satisfy condition (5).

3. Theorem 2. If the operator $T \in \mathcal{C}(\mathfrak{S}_\infty)$ with unitary spectrum possesses an eigenchain separating the spectrum, then its characteristic function $\theta_T(\lambda)$ admits the multiplicative representation

$$\theta_T(\lambda) = \left(\theta_T^*(0)\right)^{-1} \int_{\mathfrak{P}}^{\leftarrow} \left(I + \frac{H^{1/2}\, dP\, (I - PHP)^{-1} H^{1/2}}{\lambda e^{i\varphi(P)} - 1}\right), \qquad (6)$$

where $\varphi(P)$ and $\mathfrak{P}$ are the function and the maximal chain from the corresponding representations (1), (2), and $H = I - T^*T$.

In this theorem the multiplicative integral (6) is understood as the limit in operator norm (in Shatunovskii' s sense) of products of the form

$$\Pi_n = \left(I + \frac{H^{1/2}\Delta P_n\,(I - P_{nHP}n)^{-1}H^{1/2}}{\lambda e^{i\varphi(P_n)} - 1}\right)\cdots\left(I + \frac{H^{1/2}\Delta P_1\,(I - P_1 H P_1)^{-1}H^{1/2}}{\lambda e^{i\varphi(P_1)} - 1}\right),$$

where $P_j \in \mathfrak{P},\ 0 = P_0 < P_1 < \cdots < P_n = I,\ \Delta P_j = P_j - P_{j-1}$.

The multiplicative representation (6), in its structure and in its derivation, is considerably more complicated than the multiplicative representation of the characteristic operator-function of bounded operators with real spectrum and a completely continuous imaginary component, which was obtained by M. S. Brodskii (13). Moreover, the latter can be obtained as a consequence of representation (6). At the same time, the authors consider it their duty to note that the ingenious and exceptional, in their elegance, constructions of M. S. Brodskii served for the authors as a model without which they would hardly have obtained their result in such a general form.

A number of formal transformations makes it possible to obtain from representation (6) also the following:

$$\theta_T(\lambda) = U \int_{\mathfrak{P}}^{\leftarrow} \left(I + \frac{e^{i\varphi(P)} + \lambda}{e^{i\varphi(P)} - \lambda} R^*(P) H^{1/2}\, dP\, H^{1/2} R(P)\right), \tag{7}$$

where U is a unitary operator mapping $\mathfrak{K}_T$ onto $\mathfrak{K}_{T^*}$, and

$$R(P) = \int_{\mathfrak{P}_P}^{\leftarrow} \left(I + \frac{1}{2} H^{1/2} (I - PHP)^{-1}\, dP\, H^{1/2}\right), \tag{8}$$

where by $\mathfrak{P}_P$ is denoted the part of the chain $\mathfrak{P}$ from 0 to P.

A rigorous justification of representation (7) has so far been obtained for the case $T \in \mathfrak{C}(\mathfrak{S}_1)$. Apparently this representation is valid also in the general case. Under the condition $T \in \mathfrak{C}(\mathfrak{S}_1)$, formulas (7) and (8) can also be written in the form

$$\theta_T(\lambda) = U \int_{\mathfrak{P}}^{\leftarrow} \exp\left(\frac{1}{2}\frac{e^{i\varphi(P)} + \lambda}{e^{i\varphi(P)} - \lambda} R^*(P) H^{1/2}\, dP\, H^{1/2} R(P)\right), \tag{9}$$

where

$$R(P) = \int_{\mathfrak{P}_P}^{\leftarrow} \exp\left(\frac{1}{2} H^{1/2} (I - PHP)^{-1}\, dP\, H^{1/2}\right). \tag{10}$$

In an analogous form one may also rewrite formula (6). Let us explain that formula (10) is understood by us in the sense that $R(P)$ is the limit (in the sense of Shatunovskii) in operator norm of products of the form

$$\exp\left({}^1/_2 H^{1/2}(I - P_{nHP}n)^{-1}\Delta P_{nH}^{1/2}\right) \cdots \exp\left({}^1/_2 H^{1/2}(I - P_1 H P_1)^{-1}\Delta P_1 H^{1/2}\right),$$

where $P_j \in \mathfrak{P}$, $0 = P_0 < P_1 < \cdots < P_n = P$, $\Delta P_j = P_j - P_{j-1}$. Formula (9) is interpreted similarly.

As was already noted in Sec. 2, the results on the characteristic operator-function generalize to arbitrary operators $T \in \mathfrak{U}(\mathfrak{S}_\infty)$ with $|T| \lesseqgtr 1$.

Representation (9), in combination with Proposition B, as well as the corresponding generalizations of them to the case where $|T| > 1$, contain as consequences the main part of the fundamental results of V. P. Potapov [14] and Yu. P. Ginzburg [15] on the representation of J-contractive matrices and operator-functions holomorphic inside the unit circle.

Institute of Mathematics with Computing Center
Academy of Sciences of the Moldavian SSR
Odessa Civil Engineering Institute

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[1] I. Ts. Gokhberg, M. G. Krein, *Acta Sci. Math. Szeged*, **25**, 1-2 (1964).
[2] V. I. Matsaev, DAN, **154**, No. 5 (1964).
[3] Yu. I. Lyubich, V. I. Matsaev, *Matem. sborn.*, **56**, 4 (1962).
[4] B. Sz.-Nagy, C. Foiaş, *Acta Sci. Math. Szeged.*, **25**, 1-2 (1964).
[5] M. S. Brodskii, DAN, **133**, No. 6 (1960).
[6] M. S. Livshits, *Matem. sborn.*, **19**, 2 (1964).
[7] M. S. Livshits, V. P. Potapov, DAN, **72**, No. 4 (1950).
[8] Yu. L. Shmul' yan, DAN, **87**, No. 1 (1952).
[9] A. V. Shtraus, *Izv. AN SSSR, seriya matem.*, **24**, 43 (1960).
[10] V. T. Polyatskii, *Reduction to triangular form of certain nonunitary operators*, Dissertation, Odessa, 1959.
[11] M. S. Livshits, *Matem. sborn.*, **34**, 45 (1954).
[12] M. S. Brodskii, *Matem. sborn.*, **39**, 2 (1956).
[13] M. S. Brodskii, DAN, **138**, No. 3 (1961).

[14] V. P. Potapov, *Tr. Mosk. matem. obshch.*, **4**, 125 (1955).
[15] Yu. P. Ginzburg, DAN, **117**, No. 2 (1957).

Note: Figure translations are in progress. See original paper for figures.