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Abstract

Full Text

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ON TWO PROBLEMS OF MARDEŠIĆ

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MATHEMATICS

1. In this paper two examples are constructed.

Example 1. A locally connected continuum* X^*, connected by ordered continua, of dimension $\operatorname{Ind} X^* = 1$, which is not a continuous image of any ordered continuum.

Example 2. A locally connected continuum X^{**}, connected by ordered continua, for which $\dim X^{**} = 1$, while $\operatorname{ind} X^{**} = \operatorname{Ind} X^{**} = 2$.

Remark 1. These examples give an answer to two questions of Mardešić from paper (2) (problems 15 and 17).

Remark 2. In the construction of Example 2 one remarkable idea from a paper of O. V. Lokutsievskii (3) is used.

2. We proceed to the construction of the examples.

A. Canonical decomposition of a Cantor perfect set C. If a point x of the Cantor perfect set C is an endpoint of a complementary interval, then we identify this point x with the other endpoint of the same complementary interval. Every other point x of the Cantor perfect set is, by definition, a one-point element of the decomposition. The indicated decomposition of the Cantor perfect set is called the canonical one of zero rank. Obviously, this decomposition is continuous, and its space Z_0 is homeomorphic to an interval.

By a complementary interval of rank n, $n = 1, 2, ...$, to the Cantor set C we shall mean any complementary interval of length $1/3^n$.

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* A connected bicompactum is called a continuum. An ordered bicompactum (in particular, a continuum) is an ordered set which in its order topology is a bicompactum (continuum). A bicompactum X is called connected by ordered continua if for any two of its points x_1 and x_2 there exists an ordered continuum X_0, topologically contained in X, such that x_1 is its first point and x_2 is the last point in the order topology of this continuum X_0. Obviously, connectedness by ordered continua implies connectedness. If a topological space is represented as a sum of pairwise nonintersecting (disjoint) closed sets A_α, then one says

that a decomposition $\{A_\alpha\}$ of the space X is given. A decomposition $\{A_\alpha\}$ of a topological space X is called continuous (see (^{4})) if, for every element A_0 of the decomposition $\{A_\alpha\}$ and every neighborhood UA_0 of it, there is a marked neighborhood $U'A_0 \subseteq UA_0$ of this element A_0; here a set $M \subset X$ is called marked for the decomposition $\{A_\alpha\}$ if it is a sum of some elements of this decomposition. If a decomposition $\{A_\alpha\}$ of a space X is given, then the mapping f which assigns to each point $x \in X$ the element of the decomposition containing it is called the natural mapping of the space X onto the set of all elements A_α of the decomposition $\{A_\alpha\}$. A topology is introduced on this set: a set $\mathfrak{A}$ of elements of the decomposition $\{A_\alpha\}$ is considered open if the set $\bigcup_{A_\alpha \in \mathfrak{A}} A_\alpha$ is open in the space X. It is known (see (^{4})) that the space Z of a continuous decomposition $\{A_\alpha\}$ of a bicompactum X will also be a bicompactum.

The P. S. Aleksandrov line, or transfinite line: to each ordinal number $\alpha < \omega_1$ we put in correspondence a copy I_α of the interval $[0, 1]$, identifying its left endpoint with α and its right endpoint with $\alpha + 1$. The set obtained by adjoining to the indicated number ω_1 the generalized segment P is topologized by establishing the order relation. For distinct $\xi_1, \xi_2 \in P \setminus (\omega_1)$ we put $\xi_1 < \xi_2$ if one of the following conditions is fulfilled: 1) $\xi_1 \in I_{\alpha_1}$, $\xi_2 \in I_{\alpha_2}$, where $\alpha_1 < \alpha_2$; 2) $\xi_1, \xi_2 \in I_\alpha$, and on this interval $\xi_1 < \xi_2$. In addition, by definition, $\omega_1 > \xi$ for every $\xi \in P \setminus (\omega_1)$. It is easy to see that P is a continuum. This continuum P is also called the P. S. Aleksandrov line, or the transfinite line.

If in the Cantor set C we identify the endpoints of adjacent intervals of rank $> n$, $n = 1, 2, ...$, regarding all other points of C as one-point elements of the partition, then we obtain the canonical partition of rank n. The canonical partition of any rank n is continuous, and its space Z_n is homeomorphic to the discrete sum of 2^n intervals.

B. Construction of the bicompactum X^*, example 1. Consider the bicompactum $R = P \times C$. The desired bicompactum X^* is obtained from R as the space of a certain continuous partition D. We define the partition D on each set of the form $C_\xi = (\xi \times C) \subseteq R$, where ξ is a point of the bicompactum P. Let ξ be a transfinite number. Then on the set C_ξ we define our partition as the canonical partition of zero rank of the Cantor set $C_\xi \equiv C$. Let ξ be a dyadic-rational number of an interval I_α, say $\xi = m/2^k$ (m odd). Then on the set C_ξ we define the partition D as the canonical partition of rank $k-1$. If ξ is a dyadic-irrational number of some interval I_α, then on the set C_ξ every point $(\xi, x) \in C_\xi$ is a (one-point) element of the partition D. The partition space D is, by definition, our bicompactum X^*. We prove successively the properties of the space X^*.

1°. *The partition D of the bicompactum R is continuous, and the space of this partition is a locally connected bicompactum.*

Proof. Let A_0 be an arbitrary element of the partition D, and let UA_0 be an arbitrary neighborhood of it in the bicompactum R. We take a neighborhood U_1A_0 of the same element of the partition, contained in UA_0, of the form $U_1A_0 = U_1 \times V_1$, where U_1 is an interval of the ordered space P, and V_1 is open in C.

Denote by π_P the projection of the bicompactum R onto the bicompactum P, and by π_C the projection of the bicompactum R onto C. Further, denote by f_k the natural mapping of the set C onto the space Z_k of its canonical partition of rank k. By f_0 we denote the natural mapping of C onto the space Z_0 of its canonical partition of zero rank. Transfinite numbers $\xi \in P$ and the points $1/2$ of the intervals I_α we shall call points of zero rank, and dyadic-rational points $\xi = m/2^k$ (m not divisible by 2) of the intervals I_α we shall call points of rank $k-1$. Further, all dyadic-irrational points ξ of the intervals I_α we shall call points of infinite rank. Now take, in the neighborhood U_1 of the point $\pi_{PA}0$ in P, a point $\xi = m/2^k$ of least rank $k-1$. Further, take, in the neighborhood V_1 of the element $\pi_{CA}0$ of the canonical partition of rank $k-1$, a neighborhood $V_2 \subseteq V_1$ of the same element $\pi_{CA}0$, marked for this canonical partition, in such a way that the set $f_{k-1}V_2$ is connected in the space Z_{k-1}. Such a neighborhood V_2 of the element $\pi_{CA}0$ can be found, since the canonical partition of rank $k-1$ is continuous, and the space of this partition is locally connected. Consider the neighborhood $U_2A_0 = U_1 \times V_2$ of the element A_0 of the partition D of the bicompactum R; it is easily checked that it is marked for the partition D, whence it follows that the partition D is continuous.

The space X^* of the partition D is a bicompactum; we prove its local connectedness. For this it is enough to show that the image, under the natural mapping $f : R \to X^*$, of the neighborhood U_2A_0 constructed above will be a connected neighborhood fU_2A_0 of the point A_0 in the bicompactum X^*.

Consider the set $C_\xi = (\xi \times C)$, $\xi = m/2^k$. We shall first show that $f(C_\xi \cap U_2A_0)$ is connected. For this, in turn, it is enough to prove that the set $f(C_\xi \cap U_2A_0)$ is homeomorphic to the set $f_{k-1}(C_\xi \cap U_2A_0)$. But this follows from the fact that on the set $C_\xi \equiv (\xi \times C)$ our partition of the bicompactum R is precisely the canonical partition of rank $k-1$ of the set $C_\xi \equiv C$, and the neighborhood $U_2A_0 = U_1 \times V_2$ has been chosen so that V_2 is marked for this partition, and the set $f_{k-1}V_2$ is connected in $Z_{k-1} \subseteq X^*$. Consequently, the set $f(C_\xi \cap U_2A_0) = f_{k-1}(C_\xi \cap U_2A_0)$ is connected. We now prove the connectedness of the set fU_2A_0, which is open in X^*. Suppose that there is a subset $V \subset fU_2A_0$, open-and-closed in fU_2A_0. Since $f(C_\xi \cap U_2A_0)$ is connected, it is necessary···

$f(C_\xi \cap U_2A_0) \subseteq V$. Then $f^{-1}V$ is a (proper) clopen subset of the set U_2A_0, and $C_\xi \cap U_2A_0 \subset f^{-1}V$. Then the set $\pi_P(U_2A_0 \setminus f^{-1}V)$ is a proper subset of the connected neighborhood U_1 in P and is clopen in U_1, which contradicts the connectedness of U_1. Assertion 1^0 is proved.

2^0. $\dim X^* = \operatorname{ind} X^* = 1$.

It remains to prove the inequality $\operatorname{ind} X^* \leq 1$. For this it is only necessary to choose the neighborhood U_2A_0 of the element A_0 of the decomposition D, constructed in 1^0, in such a way that in the neighborhood U_1 the ends are dyadic irrational, i.e. points of infinite rank, while the two endpoints of the neighborhood V_2 in C are ends of some adjacent intervals of the set C (generally speaking, of different ranks). Then $f \operatorname{fr} U_2A_0 = \operatorname{fr} fU_2A_0$ and $\operatorname{ind} \operatorname{fr} fU_2A_0 \leq 0$.

3^0. *The bicompactum X^0 is connected by ordered continua.* Let A_1 and A_2 be two distinct points of the bicompactum X^*. Consider the sets $A_1 \subseteq R$, $A_2 \subseteq R$, and the points $\xi_1 = \pi_{PA}1$, $\xi_2 = \pi_{PA}2$ of the continuum P. Let $\xi_2 > \xi_1$. Further, let ξ_0 be the first transfinite number following ξ_2. Consider the ordered continuum $F_1 = f((\xi_1, x_1) \cup (U_1 \times x_1))$, where x_1 is any one point of the set $\pi_C \xi_1$, and U_1 is the half-interval $(\xi_1, \xi_0]$ of the ordered continuum P. The first point of this ordered continuum will be the point A_1, and the last will be the point $f(\xi_0, x_1)$. Next, on the segment fC_{ξ_0} consider the smaller segment F_2, whose first point will be the point $f(\xi_0, x_1)$, and whose last will be the point $f(\xi_0, x_2)$, where x_2 is any one (fixed) point of the set $\pi_C \xi_2$. Finally, consider the ordered continuum $F_3 = f((\xi_0, x_2) \cup (U_2 \times x_2))$, where x_2 is the same fixed point of the set $\pi_C \xi_2$, and U_2 is the half-interval $[\xi_2, \xi_0)$ of the ordered continuum P. The first point of F_3 will be $f(\xi_0, x_2)$, and the last $A_2 = f(\xi_2, x_2)$. Then $F_0 = F_1 \cup F_2 \cup F_3$ is an ordered continuum whose first point is A_1, and whose last is A_2.

We pass to the proof of the main assertion.

4^0. *The bicompactum X^* is not a continuous image of any ordered continuum.*

Proof. Suppose the contrary: let there be an ordered continuum T and a continuous mapping $g : T \to X^*$. Consider the auxiliary bicompactum $L = P \times [0, 1]$. We shall prove that the bicompactum X^* is mapped continuously onto the bicompactum L. Indeed, the bicompactum L is homeomorphic to the space of the following decomposition of the bicompactum R: on each set $C_\xi = \xi \times C$, where ξ is an arbitrary point of P, we define the auxiliary decomposition Δ of the bicompactum R as the canonical decomposition of null rank. Now it is obvious that there exists a continuous mapping h of the bicompactum X^* onto the space $Z = L$ of the decomposition Δ. The continuous mapping hg maps the ordered continuum T onto the bicompactum $L = P \times [0, 1]$. The following theorem is known (see (^{1})): if there exists a continuous mapping of an ordered continuum T onto the product of two locally connected continua (in our case L is the product of the locally connected continua P and $[0, 1]$), then each of the continua (in our case the continua P and $[0, 1]$) must have Suslin's property.* But the continuum P does not have Suslin’ s property. The contradiction obtained proves assertion 4^0.

3. The bicompactum X^{**} of Example 2 will be constructed by literally repeating the arguments and constructions of work (^{3}).

In the bicompactum X^* consider the set $F = fC_{\omega_1}$, homeomorphic to the interval $[0, 1]$. The set of one-sided points of the Cantor set of the perfect set $\omega_1 \times C$ will be denoted by $C^0_{\omega_1}$, and $fC^0_{\omega_1}$ by M. The set M, evidently, is everywhere dense in F. Consider $Q = X^*_1 \cup X^*_2$, where X^*_1, X^*_2 are nonintersecting copies of the space X^*. By F_i, M_i we shall denote—

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* A bicompactum X has Suslin’ s property if every system of pairwise disjoint open sets in it is at most countable.

denote the sets lying in X_i^*, $i = 1, 2$, and corresponding to the prescribed sets F, M in X^*. Choose on F_2 an arbitrary everywhere dense set N of the same order type as M_1 and having no points in common with M_2, except for the endpoints of F_2. As is known (see (⁵)), there exists a similarity mapping g of the segment F_2 onto F_1 which carries N into M_1. One may regard g as a continuous mapping of Q onto a certain bicompactum $X^{**} = gQ$. Obviously, $g(X_1^*) = S_1$, $g(X_2^*) = S_2$ are homeomorphisms, $X^{**} = S_1 \cup S_2$. If $E = gF_1 = gF_2$, then $S_1 \cap S_2 = E$. The bicompactum X^{**} is the desired one, i.e., it is locally connected, connected by ordered continua, has dimension $\dim X^{**} = 1$, and $\operatorname{ind} X^{**} = \operatorname{Ind} X^{**} = 2$. Moreover, $\operatorname{ind} X^{**} \neq \max\{\operatorname{ind} S_1, \operatorname{ind} S_2\}$. The fact that the bicompactum is locally connected and connected by ordered continua is proved in exactly the same way as the analogous assertions for the bicompactum X^* of Example 1. (For the proof that $\dim X^{**} = 1$, $\operatorname{ind} X^{**} = \operatorname{Ind} X^{**} = 2$, see (³).)

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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