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G. S. LITVINCHUK

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Abstract

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G. S. LITVINCHUK

NOETHER THEOREMS FOR ONE CLASS OF SINGULAR INTEGRAL EQUATIONS WITH SHIFT AND CONJUGATION

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In the present article we find conditions for normal solvability and compute the index of the functional equation

$$\sum_{k=0}^{n-1}\left\{a_k(t)\varphi[\alpha_k(t)]+b_k(t)\overline{\varphi[\alpha_k(t)]}+\frac{c_k(t)}{\pi i}\int_L\frac{\varphi(\tau)\,d\tau}{\tau-\alpha_k(t)}+\right. \tag{1}$$

$$\left.+d_k(t)\overline{\frac{1}{\pi i}\int_L\frac{\varphi(\tau)\,d\tau}{\tau-\alpha_k(t)}}+\int_L M_k[\alpha_k(t),\tau]\varphi(\tau)\,d\tau+\overline{\int_L N_k[\alpha_k(t),\tau]\varphi(\tau)\,d\tau}\right\}=g(t).$$

Here $\alpha(t)\equiv\alpha_1(t)$ is a homeomorphic mapping of the simple closed Lyapunov contour L onto itself; $\alpha'(t)\neq 0$; $\alpha_k(t)\equiv\alpha[\alpha_{k-1}(t)]$ denotes the k-th iterate of $\alpha(t)$; $\alpha_0(t)\equiv t$. The mapping $\alpha(t)$ satisfies the generalized Carleman condition $\alpha_n(t)\equiv t$. The coefficients $a_k(t)$, $b_k(t)$, $c_k(t)$, $d_k(t)$, $g(t)$ and $\alpha'(t)\subset H(L)$ (Hölder condition); $M_k(t,\tau)$, $N_k(t,\tau)$ are Fredholm kernels.

Equation (1) is the most general equation with Carleman shift and conjugation: the form of equation (1) is closed with respect to these two operations. In two special cases: 1) $n=2$, $b_0(t)=b_1(t)=d_0(t)=d_1(t)=0$ and 2) $n=2$, $a_0(t)=b_1(t)=c_0(t)=d_1(t)=0$, the Noether theorems for equation (1) were established earlier: for case 1) by the author (^{1}), and for case 2) by E. G. Khasabov and the author (^{2}).*

Along with equation (1), consider the equation

$$\sum_{k=0}^{n-1}(-1)^k\left\{a_k(t)\varphi[\alpha_k(t)]+b_k(t)\overline{\varphi[\alpha_k(t)]}+\frac{c_k(t)}{\pi i}\int_L\frac{\varphi(\tau)\,d\tau}{\tau-\alpha_k(t)}+\right. \tag{2}$$

$$\left.+d_k(t)\overline{\frac{1}{\pi i}\int_L\frac{\varphi(\tau)\,d\tau}{\tau-\alpha_k(t)}}+\int_L M_k[\alpha_k(t),\tau]\varphi(\tau)\,d\tau+\overline{\int_L N_k[\alpha_k(t),\tau]\varphi(\tau)\,d\tau}\right\}=0,$$

which we shall call the accompanying equation to (1), and the system of $2n$ singular integral equations with Cauchy kernels

$$\sum_{k=0}^{n-1}\{a_k[\alpha_{n-s+1}(t)]\rho_{2k'+1}(t)+b_k[\alpha_{n-s+1}(t)]\rho_{2k'+2}(t)+$$

$$+\lambda^k\left[\frac{c_k[\alpha_{n-s+1}(t)]}{\pi i}\int_L\frac{\alpha'_{k'}(\tau)\rho_{2k'+1}(\tau)}{\alpha_{k'}(\tau)-\alpha_{k'}(t)}\,d\tau-\frac{d_k[\alpha_{n-s+1}(t)]}{\pi i}\int_L\frac{\overline{\alpha'_{k'}(\tau)}\,\tau'^2(\sigma)\rho_{2k'+2}(\tau)}{\overline{\alpha_{k'}(\tau)}-\overline{\alpha_{k'}(t)}}\,d\tau+\right.$$

$$+\int_L M_k[\alpha_{k'}(t),\alpha_{k'}(\tau)]\alpha'_{k'}(\tau)\rho_{2k'+1}(\tau)d\tau+$$

$$\left.\left.+\int_L\overline{N_k[\alpha_{k'}(t),\alpha_{k'}(\tau)]}\alpha'_{k'}(\tau)\tau'^2(\sigma)\rho_{2k'+2}(\tau)d\tau\right\}\right\}=g[\alpha_{n-s+1}(t)],\qquad(3)$$

$$\sum_{k=0}^{n-1}\left\{\overline{b_k[\alpha_{n-s+1}(t)]}\rho_{2k'+1}(t)+a_k[\alpha_{n-s+1}(t)]\rho_{2k'+2}(t)+\right.$$

$$+\lambda^{k'}\left\{\frac{\overline{d_k[\alpha_{n-s+1}(t)]}}{\pi i}\int_L\frac{\alpha'_{k'}(\tau)\rho_{2k'+1}(\tau)}{\alpha_{k'}(\tau)-\alpha_{k'}(t)}\,d\tau-\frac{\overline{c_k[\alpha_{n-s+1}(t)]}}{\pi i}\int_L\frac{\alpha'_{k'}(\tau)\tau'^2(\sigma)\rho_{2k'+2}(\tau)}{\overline{\alpha_{k'}(\tau)}-\overline{\alpha_{k'}(t)}}\,d\tau+\right.$$

$$+\int_L N_k[\alpha_{k'}(t),\alpha_{k'}(\tau)]\alpha'_{k'}(\tau)\rho_{2k'+1}(\tau)d\tau+$$

$$\left.\left.+\int_L\overline{M_k[\alpha_{k'}(t),\alpha_{k'}(\tau)]}\alpha'_{k'}(\tau)\tau'^2(\sigma)\rho_{2k'+2}(\tau)d\tau\right\}\right\}=\overline{g[\alpha_{n-s+1}(t)]},$$

$$s=1,2,\dots,n;\qquad k'=n+k-s+1.$$

Here $\lambda=+1$ or $\lambda=-1$, according as $\alpha(t)$ maps L onto itself with preservation or reversal of the orientation on L. Since, in the case of a shift $\alpha(t)$ that reverses

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* After this work had been completed, a paper by R. A. Kordzadze (^{6}) appeared, in which equation (1) is considered in the special case $b_k(t)=d_k(t)=N_k(t,\tau)=0$. Theorem 6 of paper (^{6}) is incorrect, and it should be replaced by Proposition 2 of the present paper; the value of the index indicated by Theorem 3 of paper (^{6}) is twice the true value of the index.

the orientation on L, the number n must necessarily be even, one may assume $\lambda^n = 1$.

The functional equation adjoint to equation (1) has the form

$$\sum_{k=0}^{n-1} \lambda^k \Bigg\{ a_k[\alpha_{n-k}(t)]\alpha'_{n-k}(t)\psi[\alpha_{n-k}(t)] + \overline{b_k[\alpha_{n-k}(t)]}\alpha'_{n-k}(t)\tau'^2(s)\psi[\alpha_{n-k}(t)] - \tag{4}$$

$$-\frac{1}{\pi i}\int_L \frac{c_k[\alpha_{n-k}(\tau)]\alpha'_{n-k}(\tau)\varphi[\alpha_{n-k}(\tau)]}{\tau - t}\,d\tau -$$

$$-\frac{1}{\pi i}\int_L \frac{\overline{d_k[\alpha_{n-k}(\tau)]}\alpha'_{n-k}(\tau)\tau'^2(\sigma)\psi[\alpha_{n-k}(\tau)]}{\tau - t}\,d\tau + \int_L M_k(\tau,t)\alpha'_{n-k}(\tau)\psi[\alpha_{n-k}(\tau)]d\tau +$$

$$+\int_L \overline{N_k(\tau,t)\alpha'_{n-k}(\tau)\tau'^2(\sigma)\psi[\alpha_{n-k}(\tau)]}\,d\tau \Bigg\} = 0.$$

Let us write out, corresponding to equation (4), the system of singular equations adjoint to system (3):

$$\sum_{k=0}^{n-1} \Bigg\{ a_k[\alpha_{n-k+s-1}(t)]\omega_{2k'+1}(t) + \overline{b_k[\alpha_{n-k+s-1}(t)]}\omega_{2k'+2}(t) -$$

$$-\lambda^{s-1}\alpha'_{s-1}(t)\Bigg\{ \frac{1}{\pi i}\int_L \frac{c_k[\alpha_{n-k+s-1}(\tau)]\omega_{2k'+1}(\tau) + d_k[\alpha_{n-k+s-1}(\tau)]\omega_{2k'+2}(\tau)}{\alpha_{s-1}(\tau) - \alpha_{s-1}(t)}\,d\tau +$$

$$+\int_L M_k[\alpha_{s-1}(\tau),\alpha_{s-1}(t)]\omega_{2k'+1}(\tau)d\tau + \int_L N_k[\alpha_{s-1}(\tau),\alpha_{s-1}(t)]\omega_{2k'+2}(\tau)d\tau \Bigg\}\Bigg\} = 0,$$

$$\sum_{k=0}^{n-1} \Bigg\{ b_k[\alpha_{n-k+s-1}(t)]\omega_{2k'+1}(t) + \overline{a_k[\alpha_{n-k+s-1}(t)]}\omega_{2k'+2}(t) + \tag{5}$$

$$+\lambda^{s-1}\alpha'_{s-1}(t)\Bigg\{ \frac{1}{\pi i}\int_L \frac{d_k[\alpha_{n-k+s-1}(\tau)]\omega_{2k'+1}(\tau) + \overline{c_k[\alpha_{n-k+s-1}(\tau)]}\omega_{2k'+2}(\tau)}{\alpha_{s-1}(\tau) - \alpha_{s-1}(t)}\,d\tau +$$

$$+\int\limits_L \overline{N_k[\alpha_{s-1}(\tau),\alpha_{s-1}(t)]}\omega_{2k'+1}(\tau)d\tau+\int\limits_L \overline{M_k[\alpha_{s-1}(\tau),\alpha_{s-1}(t)]}\omega_{2k'+2}(\tau)d\tau\Big\}\Big\}=0,$$

$$s=1,2,\dots,n;\qquad k'=n+k-s+1.$$

Let l and l^* be the numbers of linearly independent solutions of the homogeneous systems (3) and (5); l_1 and l_1^* the numbers of linearly independent solutions of the homogeneous equations (1) and (4); l_2 and l_2^* the numbers of linearly independent solutions of the homogeneous equations associated with equations (1) and (4). Linear independence of solutions for systems (3) and (5) is understood, as usual, in the sense of a combination with complex coefficients, and for equations (1), (4) and their associated equations—in the sense of a combination with real coefficients.

Denote

$$\Delta_n(t)=$$

$$\det\left\|\begin{matrix} \frac{a_0-c_0}{\overline{b}_0-\overline{d}_0} & \frac{b_0+d_0}{\overline{a}_0+\overline{c}_0} & \frac{a_1-\lambda c_1}{\overline{b}_1-\lambda\overline{d}_1} & \dots & \frac{b_{n-1}+\lambda d_{n-1}}{\overline{a}_{n-1}-\lambda\overline{c}_{n-1}} \\ \frac{a_1(\alpha_{n-1})-c_1(\alpha_{n-1})}{\overline{b_1(\alpha_{n-1})}-\overline{d_1(\alpha_{n-1})}} & \frac{b_1(\alpha_{n-1})+d_1(\alpha_{n-1})}{\overline{a_1(\alpha_{n-1})}+\overline{c_1(\alpha_{n-1})}} & \frac{a_2(\alpha_{n-1})-\lambda c_2(\alpha_{n-1})}{\overline{b_2(\alpha_{n-1})}-\lambda\overline{d_2(\alpha_{n-1})}} & \dots & \frac{b_0(\alpha_{n-1})+\lambda d_0(\alpha_{n-1})}{\overline{a_0(\alpha_{n-1})}+\lambda\overline{c_0(\alpha_{n-1})}} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \dots & \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{a_{n-1}(\alpha)-c_{n-1}(\alpha)}{\overline{b_{n-1}(\alpha)}-\overline{d_{n-1}(\alpha)}} & \frac{b_{n-1}(\alpha)+d_{n-1}(\alpha)}{\overline{a_{n-1}(\alpha)}+\overline{c_{n-1}(\alpha)}} & \frac{a_0(\alpha)-\lambda c_0(\alpha)}{\overline{b_0(\alpha)}-\lambda\overline{d_0(\alpha)}} & \dots & \frac{b_{n-2}(\alpha)+\lambda d_{n-2}(\alpha)}{\overline{a_{n-2}(\alpha)}+\lambda\overline{c_{n-2}(\alpha)}} \end{matrix}\right\|.$$

It is easy to see that the matrix written above is the principal matrix of the system of singular integral equations (3).

Proposition 1. For the solvability of the nonhomogeneous equation (1) it is necessary and sufficient that the nonhomogeneous system of equations (3) be solvable. If equation (1) is solvable and $\varphi(t)$ is its solution, then the vector $\rho(t)$ with components

$$\rho_{2k+1}(t)=\varphi[\alpha_k(t)],\qquad \rho_{2k+2}(t)=\overline{\varphi[\alpha_k(t)]},\qquad k=0,1,\dots,n-1,$$

is a solution of system (3). Conversely, if system (3) is solvable and the vector $\rho(t)=\{\rho_1(t),\rho_2(t),\dots,\rho_{2n-1}(t),\rho_{2n}(t)\}$ is its solution, then the function

$$\varphi(t)=\frac{1}{2n}\sum_{k=0}^{n-1}\left\{\rho_{2n-2k+1}[\alpha_k(t)]+\overline{\rho_{2n-2k+2}[\alpha_k(t)]}\right\}$$

is a solution of equation (1).

Proposition 2. $l = l_1 + l_2, \quad l^* = l_1^* + l_2^*$.

Proposition 3. A fundamental system of solutions of the system of singular equations (3) can be chosen in such a way that l_1 solutions satisfy the condition

$$\rho_{2k+1}(t) = \rho_1[\alpha_k(t)], \qquad \rho_{2k+2}(t) = \overline{\rho_1[\alpha_k(t)]}, \qquad k = 0, 1, \dots, n-1,$$

and the remaining l_2 solutions satisfy the condition

$$\rho_{2k+1}(t) = (-1)^k \rho_1[\alpha_k(t)], \qquad \rho_{2k+2}(t) = (-1)^k \overline{\rho_1[\alpha_k(t)]},$$

$$k = 0, 1, \dots, n-1.$$

Analogously, a fundamental system of solutions of the adjoint system (5) may be selected so that l_1^* of its solutions satisfy the condition

$$\omega_{2k+1}(t) = \lambda^k \alpha'_{n-k}(t) \omega_1[\alpha_{n-k}(t)], \qquad \omega_{2k+2}(t) = \overline{\lambda^k \alpha'_{n-k}(t) t'^2(s) \omega_1[\alpha_{n-k}(t)]},$$

$$k = 0, 1, \dots, n-1,$$

and l_2^* solutions satisfy the condition

$$\omega_{2k+1}(t) = (-1)^k \lambda^k \alpha'_{n-k}(t) \omega_1[\alpha_{n-k}(t)],$$

$$\omega_{2k+2}(t) = (-1)^k \overline{\lambda^k \alpha'_{n-k}(t) t'^2(s) \omega_1[\alpha_{n-k}(t)]},$$

$$k = 0, 1, \dots, n-1.$$

Proposition 4. Equation (1) and its adjoint equation (2) have one and the same index, i.e. $l_1 - l_1^* = l_2 - l_2^*$. This proposition is based on F. V. Atkinson' s theorem [3] on the stability of the index.

Using Propositions 1-4 and the known facts of the theory of systems of singular integral equations with Cauchy kernel [4], one can establish the following basic theorems.

Theorem 1. In order that the functional equation (1) be normally solvable, it is necessary and sufficient that the condition $\Delta_n(t) \neq 0$ on L be satisfied.

Theorem 2. The index of the functional equation (1) is computed by the formula $I = \operatorname{Ind} \Delta_n(t)$.

Theorem 3. For the solvability of the functional equation (1) it is necessary and sufficient that the condition

$$\operatorname{Re} \int_L g(t)\psi(t)\, dt = 0,$$

be satisfied, where $\psi(t)$ is any solution of the adjoint equation (4).

The difference that exists between the cases of even and odd n is clearly illustrated by the following simple example. Consider the equation with shift

$$a_0(t)\varphi(t) + \frac{c_1(t)}{\pi i} \int_L \frac{\varphi(\tau)}{\tau - \alpha(t)}\, d\tau = g(t). \tag{6}$$

Here

$$\Delta_n(t) = \prod_{k=0}^{n-1} |a_0[\alpha_k(t)]|^2 + \prod_{k=0}^{n-1} |c_1[\alpha_k(t)]|^2 - 2\lambda^{n/2} \operatorname{Re} \prod_{k=0}^{n-1} a_0[\alpha_k(t)]\, \overline{c_1[\alpha_k(t)]},$$

if n is even;

$$\Delta_n(t) = \prod_{k=0}^{n-1} |a_0[\alpha_k(t)]|^2 + \prod_{k=0}^{n-1} |c_1[\alpha_k(t)]|^2 + 2i \operatorname{Im} \prod_{k=0}^{n-1} a_0[\alpha_k(t)]\, \overline{c_1[\alpha_k(t)]},$$

if n is odd.

Consequently, $\operatorname{Ind} \Delta_n(t) = 0$ for even n and, generally speaking, $\operatorname{Ind} \Delta_n(t) \neq 0$ for odd n, i.e. equation (6) with even n is always a Fredholm equation, while equation (6) with odd n, generally speaking, is a Noether equation. As a "model" equation for equation (6) with even n, one may, for example, take the equation studied by the author [1,5] in equation (6) with $n = 2$. In the case of odd n, equation (6) with $n = 1$, i.e. the classical singular equation with Cauchy kernel, may serve as a "model" equation.

Odessa State University
named after I. I. Mechnikov

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Note: Figure translations are in progress. See original paper for figures.