ON TAYLOR-DIRICHLET SERIES

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Abstract

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MATHEMATICS

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ON TAYLOR-DIRICHLET SERIES

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Consider the series

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^{\alpha_n} \exp(-\lambda_n z), \tag{1}$$

whose coefficients and exponents satisfy the following conditions:

$$H > \lambda_{n+1} - \lambda_n > h > 0; \tag{2}$$

$$\alpha_{n+1} \geq \alpha_n; \tag{3}$$

$$\overline{\lim}(\lambda_n^{-1} \ln |a_n|) = 0. \tag{4}$$

Definition. A series (1) satisfying conditions (2), (3), (4) will be called a **reduced series**.

Remark. If $\overline{\lim}(\lambda_n^{-1} \ln |a_n|)$ is finite, then (4) will always be fulfilled after a suitable linear change of the variable z. The domain of convergence of a Taylor-Dirichlet series was investigated by G. L. Lunts (^{5}). We consider only the case when, for a reduced series, $\operatorname{Re} z_0 > -1$. Then, obviously, the series certainly converges in the domain $Q\{\operatorname{Re} z > 0,\ |z - z_0| < 1\}$. If the numbers α_n are not all integers, then we single out a single-valued branch of $f(z)$, for which we make in the z-plane an infinite cut from z_0 to the left.

From (2) and (4) there follows the existence of such a sequence of positive numbers $\{q_n\}$ and such an infinite sequence of indices $\{n_k\}$ that

$$|a_n| < Cq_n; \qquad |a_{n_k}| = q_{n_k}; \qquad q_{n+1}q_n^{-1} \to 1. \tag{5}$$

Such a sequence $\{q_n\}$ is constructed, for example, in (^{1,3}).

Definition. Any infinite sequence $\{n_k\}$ for which there is a $\{q_n\}$ satisfying (5) will be called a **sequence of principal indices of the reduced series**.

Theorem 1 (on gaps). Let, for a reduced series, $\operatorname{Re} z_0 > -1$, and let there be a sequence of principal indices $\{n_k\}$ and a sequence of positive numbers $\{m_k\}$, $m_k \to \infty$, such that $a_{n_k+\nu} = 0$, $\nu = 1, 2, \ldots, m_k$.

Then the sum of the reduced series $f(z)$ cannot be analytically continued from the domain Q into the domain $R\{\operatorname{Re} z < 0,\ |z - z_0| > 1\}$.

Theorem 2. Let, for a reduced series, $\operatorname{Re} z_0 > -1$; let $\{n_k\}$ be some sequence of principal indices. Put

$$a_n^* = \begin{cases} -a_n & \text{for } n = n_k, \\ a_n & \text{for } n \neq n_k; \end{cases}$$

$$f^*(z) = \sum_{n=1}^{\infty} a_n^* (z - z_0)^{\alpha_n} \exp(-\lambda_n z).$$

Let $n_{k+1} - n_k \to \infty$.

Then at least one of the two functions $f(z)$, $f^*(z)$ cannot be continued from Q into R.

Theorems 1 and 2 are proved with the aid of the following lemma.

Lemma. Suppose that the sum of the above series $f(z)$ is analytic in a closed bounded domain D having a common part with Q.

Then the family of functions $\{f_n(z)\}$, where

$$f_n(z) = \left[f(z) - \sum_{k=1}^{n} a_k (z - z_0)^{\alpha_k} \exp(-\lambda_k z) \right] q_n^{-1} (z - z_0)^{-\alpha_n} \exp(\lambda_n z),$$

is bounded in D.

This lemma is a generalization of a theorem proved by Agmon (^{1}) for Dirichlet series.

If the sequence $\{\alpha_n\}$ satisfies, in addition to (3), the condition $\alpha_{n+1} - \alpha_n \to 0$, then the boundary of the domain of convergence of the above series is the imaginary axis. In this case the following theorems on the distribution of singularities of the sum of the series $f(z)$ are valid.

Theorem 3. Suppose that for the above series the condition $\alpha_{n+1} - \alpha_n \to 0$ is fulfilled. Suppose that on a segment of the imaginary axis of length greater than $2\pi h^{-1}$, not containing the point z_0, the sum of the series has only a finite number of singular points, and all of them are poles of order not exceeding m. Let

$$d_L = \max_{z \in L} |z - z_0|;$$

let d be any number smaller than d_L.

Then

$$a_n = O\left(n^{m-1} d^{-\alpha_n}\right).$$

The proof is carried out with the aid of a generalization of the method used in (^{2}); see also (^{4}).

On the other hand, it is immediately seen that if at a point z_1 on the imaginary axis $f(z)$ has a pole of order m, then $a_n = \Omega(n^{m-1} D^{-\alpha_n})$, where D is any number greater than $|z_1 - z_0|$.

If $\alpha_n \to \infty$, then this contradicts Theorem 3 for $d > D$. Thus the following is true.

Theorem 4. If $\alpha_{n+1} - \alpha_n \to 0$, $\alpha_n \to \infty$, then on every segment of the imaginary axis of length greater than $2\pi h^{-1}$ and not containing the point z_0, the sum of the above series $f(z)$ must have at least one singular point which is not a pole.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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