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Abstract

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MECHANICS

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ON THE ENERGY-MOMENTUM TENSOR

AND ON MACROSCOPIC INTERNAL INTERACTIONS

IN A GRAVITATIONAL FIELD AND IN MATERIAL MEDIA

Let us consider a continuum forming a four-dimensional manifold G, defined by the totality of values of a system of four variables $\xi^1, \xi^2, \xi^3, \xi^4$. Individual points of the continuum M are defined by specifying three numbers ξ^1, ξ^2, ξ^3. For fixed ξ^1, ξ^2, ξ^3 and variable ξ^4, in the space G one obtains a world line corresponding to the point M. Within the framework of GTR (the general theory of relativity), according to its basic assumption, the manifold G is a four-dimensional Riemannian space. In the space G the variables ξ^i form a comoving coordinate system.

Let an observer describing the physical phenomena in the medium under consideration use some coordinate system of his own, x^1, x^2, x^3, x^4, taken in this same space G. At every point of G one may consider vectors $d\mathbf{r}$ with infinitesimal components $d\xi^i$ or dx^i, and, correspondingly, the vector bases $\mathfrak{e}_i = \partial\mathbf{r}/\partial\xi^i$ and $\mathbf{e}_i = \partial\mathbf{r}/\partial x^i$.

From the definition of Riemannian spaces it follows that the metric, parallel transport, and curvature tensor in any of the coordinate systems and, in particular, in the system x^i, are defined by equalities of the form

$$|d\mathbf{r}| = ds; \qquad ds^2 = g_{ij}dx^i dx^j; \qquad g_{ij} = (\mathbf{e}_i\mathbf{e}_j); \qquad \partial\mathbf{e}_i/\partial x^j = \Gamma^s_{ij}\mathbf{e}_s;$$

$$\Gamma^s_{ij} = \frac{1}{2}g^{sl}\left[\partial g_{il}/\partial x^j + \partial g_{jl}/\partial x^i - \partial g_{ij}/\partial x^l\right]; \qquad R_{il} = R^s_{isl};$$

$$R^s_{ijl} = \partial\Gamma^s_{il}/\partial x^j - \partial\Gamma^s_{jl}/\partial x^i + \Gamma^s_{pj}\Gamma^p_{li} - \Gamma^s_{pi}\Gamma^p_{lj}; \qquad R = g^{il}R_{il}. \tag{1}$$

Between the systems of variables x^i and ξ^i in one and the same space G there exists a finite relation

$$x^i = x^i(\xi^1, \xi^2, \xi^3, \xi^4), \tag{2}$$

which we shall call the law of motion of the medium under consideration.

Along with other variable characteristic quantities μ^k $(k = 1, 2, \ldots, N)$, the coordinates x^i, connected with ξ^i by the generally sought law of motion (2), and the components of the metric tensor $g_{ij}(x^l)$ will be regarded as independent defining parameters of the state and motion of the model of the medium. The variable defining parameters μ^k may have different geometric or physical nature —these are characteristics of various fields (for example, the electromagnetic field) and characteristics of various internal degrees of freedom. Among the parameters μ^k there may be scalars and certain complexes forming a system of components of spinors, vectors, or tensors in the four-dimensional space G.

Besides the variable defining quantities $\chi^A\{\mu^k, x^i, g_{ij}\}$, the properties and state of the medium are generally connected with the values of certain, in a definite sense constant, parameters k^B $(B = 1, 2, \ldots, N_1)$, which may characterize various geometric structural and, in general, physical properties and, in particular, properties of anisotropy (5).

The available theoretical experience and general fundamental physical methods show that, as the initial basis for constructing

for models of media one may use variational equations of the form

$$\delta \int_V L \, d\tau + \delta W + \delta W^* = 0, \tag{3}$$

where V is an arbitrary four-dimensional volume in G; $d\tau = \sqrt{-g}\, dx^1 dx^2 dx^3 dx^4$ is the element of four-dimensional volume, $g = |g_{ij}|$; L is some function of χ^A and k^B and of the partial derivatives of χ^A with respect to x^i or ξ^i. The variation δW is represented by an integral over the surface Σ bounding V, of a linear combination of $\delta\chi^A$ and their partial derivatives with respect to x^i, and is determined by specifying the function L. The variation δW^* is a prescribed functional in the form of the sum of an integral over the volume V and over the surface Σ of linear combinations composed of the variations χ^A and the variations of their partial derivatives.

The presence among the determining parameters of successive derivatives and, especially, derivatives with respect to time complicates the physical problem of specifying a model by means of thermodynamic functions (1); in this case, apparently, variational principles are the most convenient starting point. Below, for simplicity, we shall restrict ourselves to the case when $\delta W^* = 0$*, and take equation (3) in the form:

$$\delta \int_V \left(\frac{1}{2\varkappa} R + \Lambda \right) d\tau + \delta W = 0, \tag{4}$$

where $\varkappa = 8\pi k / c^4 \approx 2 \cdot 10^{-48}\ \mathrm{sec}^2/\mathrm{cm} \cdot \mathrm{g}$; R and Λ are scalars, with**

$$\Lambda = \Lambda(x^i, \partial x^i / \partial \xi^j = x^i{}_j, g_{ij}, \mu^k, \nabla_j \mu^k).$$

In computing the variations in (4) it is necessary to take into account the relations
$\delta x^i = x'^i(\xi^1, \xi^2, \xi^3, \xi^4) - x^i(\xi^1, \xi^2, \xi^3, \xi^4)$. The infinitesimal quantities δx^i may be regarded as the components of a vector.

Variations at constant x^i will be denoted by the symbol ∂. For example:
$\partial g_{ij} = g'_{ij}(x^l) - g_{ij}(x^l) = \partial^* g_{ij} + \nabla_i \partial \eta_j + \nabla_j \partial \eta_i$, where $\partial\eta_1, \partial\eta_2, \partial\eta_3, \partial\eta_4$ are infinitesimal functions of x^i; ∂g_{ij} is the variation of the metric tensor caused by deformation of the reference system, by curvature, and by the variation of the curvature of the space G. It is obvious that in the special theory of relativity (s.t.r.) the variations $\partial^{**} g_{ij} = \nabla_i \partial \eta_j + \nabla_j \partial \eta_i$ may be regarded as a change of the metric of the pseudo-Euclidean space G caused by its deformation. It is easy to see that in s.t.r., when curvilinear coordinates are used, the equality $\partial g_{ij} = \partial^{**} g_{ij}$ holds.

Solutions of the system of equations $\nabla_i \partial \eta_j + \nabla_j \partial \eta_i = 0$ in s.t.r. determine infinitesimal rigid displacements $\partial \eta_i$ corresponding to the invariance of the components g_{ij}. It is obvious that the variations δx^i and $\partial \eta_i$ are, in general, completely independent.

For tensor quantities we have

$$\delta \mu^k = \partial \mu^k + \delta x^l \nabla_l \mu^k, \qquad \partial x^j{}_i = x^s{}_i \nabla_s \delta x^j - \delta x^l \nabla_l x^j{}_i. \tag{5}$$

Let us also note the following equality:

$$\delta d\tau = (\partial \sqrt{-g} / \sqrt{-g} + \nabla_i \delta x^i)\, d\tau, \quad \text{with} \quad \partial \sqrt{-g} / \sqrt{-g} = \tfrac{1}{2} g^{ij} \partial g_{ij}. \tag{6}$$

The well-known relations are (3)$'$

$$\partial R = -R^{ij} \partial g_{ij} + \nabla_l w^l, \quad \text{where} \quad \nabla_l w^l = g^{ij} \partial R_{ij}; \quad w^l = (g^{ij} \delta^l_s - g^{il} \delta^j_s) \partial \Gamma^s_{ij}. \tag{7}$$

Using (4)–(7), after the usual transformations the equations in va-

* Irreversible effects and internal sources of energy are taken into account by means of δW^*; the corresponding theory is developed in our work (2).

** The subsequent conclusions can be generalized to the case when L depends on derivatives of the determining parameters with respect to coordinates of higher order. Below the symbol ∇ acts only on tensor indices in the system x^i; in particular,
$\nabla_l x^j{}_i = \partial x^j{}_i / \partial x^l + x^j{}_s \Gamma^s_{li}$.

we bring the variations to the form:

$$\int_V \left\{ -\frac{1}{2} \left[\frac{1}{\chi} \left(R^{ij} - \frac{1}{2} g^{ij} R \right) + T^{ij} \right] \delta^* g_{ij} + \nabla_j T^{ij} \delta\eta_i + \left(\frac{\partial\Lambda}{\partial\mu^k} - \nabla_l \frac{\partial\Lambda}{\partial\nabla_l\mu^k} \right) \delta\mu^k - \right.$$

$$\left. - \left[\frac{\partial\Lambda}{\partial x^s_j} \nabla_i x^s_j + \nabla_s \left(\frac{\partial\Lambda}{\partial x^i_j} x^s_j \right) \right] \delta x^i \right\} d\tau + \delta W +$$

$$+ \int_\Sigma \left\{ - \left[\frac{1}{\chi} \left(R^{ij} - \frac{1}{2} g^{ij} R \right) + T^{ij} \right] \delta\eta_i + \left[\left(\frac{1}{2\chi} R + \Lambda \right) \delta^j_i + \frac{\partial\Lambda}{\partial x^i_s} x^j_s \right] \delta x^i \right.$$

$$\left. + \frac{\partial\Lambda}{\partial\nabla_j\mu^k} \delta\mu^k + \left(\frac{\partial\Lambda}{\partial\nabla_l\mu^k} \frac{\partial\nabla_l\mu^k}{\partial g_{pq,j}} \right) \delta g_{pq} + \frac{1}{2\chi} (g^{pq}\delta^j_s - g^{pj}\delta^q_s) \delta\Gamma^s_{pq} \right\} n_j d\sigma = 0, \tag{8}$$

where the abbreviated notation has been introduced

$$T^{ij} = -\frac{2}{\sqrt{-g}} \left[\frac{\partial\sqrt{-g}\Lambda}{\partial g_{ij}} + \frac{\partial\sqrt{-g}\Lambda}{\partial\nabla_l\mu^k} \frac{\partial\nabla_l\mu^k}{\partial g_{ij}} - \frac{\partial}{\partial x^s} \left(\sqrt{-g} \frac{\partial\Lambda}{\partial\nabla_l\mu^k} \frac{\partial\nabla_l\mu^k}{\partial g_{ij,s}} \right) \right]. \tag{9}$$

The variations δx^i, $\delta\mu^k$, $\delta\eta_i$ are arbitrary, and within the framework of GRT the variations $\delta^* g_{ij}$ are also arbitrary; putting δW and all these variations equal to zero on Σ, from (8) we obtain:

$$R^{ij} - {}^1/{}_2 g^{ij} R = -\chi T^{ij}; \tag{10}$$

$$\nabla^{ij}_{jT} = 0; \tag{11}$$

$$\frac{\partial\Lambda}{\partial x^s_j} \nabla_i x^s_j + \nabla_s \left(\frac{\partial\Lambda}{\partial x^i_j} x^s_j \right) = 0; \tag{12}$$

$$\frac{\partial\Lambda}{\partial\mu^k} - \nabla_l \frac{\partial\Lambda}{\partial\nabla_l\mu^k} = 0. \tag{13}$$

Equations (10)–(13) form a complete system of equations for $x^i(\xi^j)$, g_{ij}, and μ^k. In SRT, $\delta^* g_{ij} = 0$, and therefore equations (10) are absent. In this case the metric g_{ij} depends essentially only on four functions, and therefore equations (11)–(13) form a complete system. Thus the metric of the observer' s systems is determined, and it is obtained from the properties of the function Λ. If the reference systems x^i are specified from physical data, then this imposes corresponding restrictions on the properties of the function Λ. In particular, in the special theory these restrictions may lead to the coincidence or equivalence of the systems of equations (11) and (12), (13).

By virtue of equations (10)–(13), equation (8), for arbitrary variations, leads to the equalities

$$\delta W = \int_\Sigma \left(\Theta^{ij} \delta\eta_i + P_i^j \delta x^i + M_k^j \delta\mu^k + Q^{pqj} \delta g_{pq} + G_l^{pqj} \delta\Gamma^l_{pq} \right) n_j d\sigma, \tag{14}$$

where

$$\Theta^{ij} = T^{ij} + \frac{1}{\chi}\left(R^{ij} - \frac{1}{2} g^{ij} R \right); \tag{15}$$

$$P_i^j = -\Lambda \delta_i^j - \frac{\partial \Lambda}{\partial x_s^i} x_s^j - \frac{1}{2\chi} R \delta_i^j\ {}^*; \tag{16}$$

$$M_k^j = -\frac{\partial \Lambda}{\partial \nabla_j \mu^k}; \tag{17}$$

$$Q^{pqj} = -\frac{\partial \Lambda}{\partial \nabla_l \mu^k} \frac{\partial \nabla_l \mu^k}{\partial g_{pq,j}}; \tag{18}$$

$$G_l^{pqj} = \frac{1}{2\chi}(g^{pj}\delta_l^q - g^{pq}\delta_l^j). \tag{19}$$

* When, in (8) and (14), $\partial\mu^k$ is replaced by $\delta\mu^k$, in (16) an additional term $+\dfrac{\partial \Lambda}{\partial \nabla_j \mu^k} \nabla_i \mu^k$ will appear.

In an O.R.F., by virtue of (10), we have $\theta^{ij} = 0$, whereas in an A.R.F. $\theta^{ij} \neq 0$. The tensor $\theta^{ij} = T^{ij}$ in an A.R.F. coincides with Einstein' s energy-momentum tensor. However, in more general cases, according to (15) an additional term appears.

Equalities (15)–(19) may be regarded as generalized equations of state of the medium and the gravitational field, determining the internal stresses. These equations are analogous to the generalized Hooke' s law. As is known, when variational principles are used, in the theory of elasticity as well one can eliminate the internal stresses and construct systems of equations of the same type as equations (10)–(13).

However, the experience of the theory of elasticity shows that the explicit introduction of internal stresses and of the corresponding energy fluxes through Σ is useful and fruitful both in theory and in applications, in particular in the formulation of boundary and initial conditions. The tensor P_i^j is in general non-symmetric and differs from the symmetric tensor θ_i^j; it may be regarded as a second energy-momentum tensor, representing a generalization of the ordinary three-dimensional tensor of internal stresses.

In an A.R.F., in many important cases the equality $P_i^j = \theta_i^j$ holds. In an O.R.F. and in a generalized A.R.F., in (15) and (16) there appears the additional gravitational pressure $-\frac{1}{2\chi}R\delta_i^j$, caused by the scalar curvature R of the space G.

The energy equations corresponding to the tensors θ_i^j and P_i^j, on the basis of (11) and (12), may be written in the form

$$u^i\nabla_j T_i^j = 0, \qquad u^i\nabla_j P_i^j = -\frac{d}{ds}\left(\frac{1}{2\chi}R + \Lambda\right) + \frac{\partial\Lambda}{\partial x_j^s}u^i\nabla_i x_j^s, \tag{20}$$

where $u^i = dx^i/ds$ is the four-dimensional velocity vector, and ds is an element of arc along the world line. Equations (20) can be transformed to the three-dimensional form adopted in thermodynamics, with the introduction of specific internal energies (see ([2,4]).

The fourth-rank tensor G_l^{pqj} is determined by the metric. For an infinitesimal particle, under variation of the curvature, the variational influx of energy corresponding to this tensor is represented in the form

$$\delta E_G = \frac{\delta W_G}{ds} = \frac{1}{ds}\int_\Sigma G_l^{pqj}\delta\Gamma_{pq}^{\,l} n_j\,d\sigma = \frac{1}{ds}\int_\Sigma G^{qpj} n_j \nabla_q \delta g_{pq}\,d\sigma =$$

$$= -\frac{1}{2\chi\,ds}\nabla_{lE}^{l}\,d\tau = -\frac{1}{2\chi\,ds}g^{ij}\delta R_{ij}\,d\tau;$$

here ds is an interval—the length of the arc along world lines.

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1. L. I. Sedov, *Proceedings of the XV International Congress on Theoretical and Applied Mechanics*, Munich, 1965.
2. L. I. Sedov, *UMN*, **20**, no. 5 (1965).
3. L. D. Landau, E. M. Lifshitz, *Field Theory*, Moscow, 1963.
4. L. I. Sedov, *Applied Mathematics and Mechanics*, **29**, no. 1 (1965).
5. V. V. Lokhin, L. I. Sedov, *Applied Mathematics and Mechanics*, **27**, no. 3 (1963).

Note: Figure translations are in progress. See original paper for figures.

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