CRITICISM AND BIBLIOGRAPHY

V. I. Krylov, M. L. Rasulov

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Abstract

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M. L. Rasulov.

“The Method of the Contour Integral and Its Application to the Study of Problems for Differential Equations.” Fizmatgiz Publishing House, 1964

As is well known, the application of the Fourier method to the solution of mixed problems with separated variables t and x of the form

$$M\left(t, \frac{\partial}{\partial t}\right) u = L\left(x, \frac{\partial}{\partial x}\right) u + f(x,t),$$

$$L_{\gamma}(u) = 0 \quad \text{on } \Gamma, \tag{1}$$

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \Phi_{\nu}(x) \quad (k = 0, 1, \dots, q-1)$$

leads to the problem of expanding functions in a series in the eigenfunctions of the corresponding boundary-value problem with a parameter (spectral problem):

$$L\left(x, \frac{d}{dx}\right) v - \lambda^m v = h(x), \qquad L_{\nu}(v) = 0 \quad \text{on } \Gamma, \tag{2}$$

where $M\left(t, \frac{\partial}{\partial t}\right)$ is a linear differential expression in t of order q, and $L\left(x, \frac{\partial}{\partial x}\right)$ is a linear differential expression of order m with respect to the point x of some n-dimensional bounded domain D with boundary Γ.

When problem (2) is not self-adjoint, the question of the existence and completeness of the system of eigenfunctions and associated functions, generally speaking, has been insufficiently studied. Moreover, this system may not possess the property of orthogonality, and there arises the need to develop an exact method for solving the mixed problem (1).

Cauchy was the first to indicate the application of the integral residue to the solution of one-dimensional mixed problems of the form (1) with separated variables for equations with constant coefficients.

The main drawback of Cauchy’ s residue method consists in the unfortunate choice of the corresponding spectral problem, as a result of which the solution of the mixed problem is based on an unsuccessful formula for expanding arbitrary sufficiently smooth functions in a series of residues. In view of this, Cauchy’ s residue method proved inapplicable to the solution of mixed problems for equations with variable coefficients. Moreover, it is not clear how it can be applied to the solution of mixed problems with nonseparated variables even for equations with constant coefficients.

In connection with the solution of the principal mixed problems for the heat-conduction equation, Poincaré noted the validity of the following important expansion formula

$$h(x) = -\varepsilon_\lambda \lambda^{m-1} \int_D G(x,\xi,\lambda)\, h(\xi)\, dD_\xi, \tag{3}$$

where $G(x,\xi,\lambda)$ is the Green' s function of problem (2) for the particular case when $L\left(x,\frac{\partial}{\partial x}\right)$ is the Laplace operator, and ε_λ denotes the total residue with respect to λ.

Formula (3), on the one hand, expresses the fact of completeness of the system of residues of the solution of the corresponding spectral problem of the form (2), and, on the other hand, makes it possible to compute the expansion coefficients if the Green' s function has been constructed and its poles have been found.

In the case when $L\left(x,-\dfrac{\partial}{\partial x}\right)$ is an ordinary linear differential expression considered on a finite interval $[a,b]$, and L_ν are linear forms in the derivatives $v(a),\dots,v^{(m-1)}(a),\ v(b),\dots,\ v^{(m-1)}(b)$ (not depending on λ), the validity of formula (3) was proved by Birkhoff in 1908. In the same work, in the case of simple poles of the Green' s function, Birkhoff proved the biorthogonality of the systems of eigenfunctions of the one-dimensional spectral problem of the form (2) and of the problem adjoint to it. Thereby he justified the applicability of the Fourier method scheme to the solution of the corresponding one-dimensional mixed problem of the form (1) in the case when the Green' s function of the corresponding problem (2) has no multiple poles.

Later the validity of formula (3) was proved by Ya. D. Tamarkin in the one-dimensional problem of the form (2) for the case when, in the differential expression $L\left(x,\dfrac{\partial}{\partial x}\right)$, the coefficient of the derivative of order $m-k$ is a polynomial of degree k, and the coefficients of the linear forms L_ν are polynomials of degrees not exceeding m.

It should be said that, after the above-described works of Cauchy, Poincaré, Birkhoff, and Tamarkin, in the study of mixed problems it was first of all necessary to dwell on the clarification of the following questions.

1) The representability of a sufficiently smooth solution of a mixed problem of the form (1) for equations with variable coefficients in the form of an integral residue in the case when problem (2) is not self-adjoint and has a multiple spectrum.

2) The representability of a sufficiently smooth solution of a mixed problem for systems of equations with variable coefficients (generally speaking, discontinuous ones) with nonseparable variables (here the equations, generally speaking, may contain mixed derivatives, and the boundary conditions derivatives with respect to time) in the form of an integral residue.

3) To prove the existence of a solution of the mixed problems under consideration and to investigate the conditions for their well-posedness. As far as possible, to obtain necessary and sufficient solvability conditions for these problems.

The book under review by M. L. Rasulov is devoted to a systematic exposition of two methods of the author—the residue method and the contour integral method—as well as to the application of these methods to the study of the problems listed above, 1)–3).

The book consists of an introduction and two parts:

1. Residue method.

2. Contour integral method.

The introduction gives a survey of results of other authors connected with the author' s results set forth in the book, as well as a survey of the author' s results set forth in the book by sections.

The first part of the book consists of five chapters.

In the first two chapters the results of Ya. D. Tamarkin on the asymptotic representation of solutions of ordinary linear equations depending on a complex parameter are set forth; these are used by the author in the subsequent chapters. The third chapter is devoted to the proof of the validity of the basic expansion formulas.

In § 1 of Ch. III, the spectral problem for systems of equations with discontinuous coefficients is considered:

$$-\frac{dy^{(i)}}{dx} - a^{(i)}(x,\lambda)y^{(i)} = f^{(i)}(x), \quad x \in (a_i, b_i), \quad (i = 1, \dots, n) \tag{4}$$

$$\sum_{i=1}^{n} \{\alpha^{(i)}(\lambda)y^{(i)}(a_i,\lambda) + \beta^{(i)}(\lambda)y^{(i)}(b_i,\lambda)\} = 0, \tag{5}$$

where

$$a^{(i)}(x,\lambda) = \lambda a^{(i)}(x) + \sum_{\nu=0}^{Q} \lambda^{-\nu} a_{\nu}^{(i)}(x), \tag{6}$$

and $a^{(i)}(x)$, $a_\nu^{(i)}(x)$ are square matrices of order r, $\alpha^{(i)}(\lambda)$, $\beta^{(i)}(\lambda)$ are polynomial matrices in λ, of size $nr \times r$, and (a_i, b_i) are mutually nonoverlapping intervals having common endpoints.

It is proved that if $a^{(i)}(x)$, $a_\nu^{(i)}(x)$ are sufficiently smooth on the interval $[a_i, b_i]$, condition (5) is regular, and the roots of the characteristic equation

$$\det(a^{(i)}(x) - \theta E) = 0$$

are simple, while the arguments of these roots and of their differences do not depend on x, then there exists a sequence of expanding closed contours Γ_ν such that

$$-\frac{1}{2\pi\sqrt{-1}}\lim_{\nu\to\infty}\int_{\Gamma_\nu} y^{(i)}(x,\lambda,f)\,d\lambda = \left(a^{(i)}(x)\right)^{-1} f^{(i)}(x), \tag{7}$$

for every vector-function $f^{(i)}(x) \in L_2(a_i, b_i)$, where convergence is understood in the sense of the metric of $L_2(a_i, b_i)$ (see Theorem 8 on p. 136).

In addition, in § 1 of Chapter III a theorem is proved on the asymptotic representation of the zeros of the characteristic determinant of the Green matrix of problem (4), (5), and on a lower estimate for this determinant outside a certain neighborhood of the poles (see Theorem 7 on p. 112). In particular, for $n = 1$, if the coefficients of system (4) do not contain negative powers of λ, $a^{(i)}(x) = 1$, and the boundary condition (5) does not depend on the parameter λ, the validity of formula (7) was established in a joint work of Birkhoff and Langer. In this case the left-hand side of (7) coincides with a number of terms in the expansion of the vector-function $f^{(i)}(x)$ in a series in the eigenfunctions and associated functions of the corresponding problem (4), (5).

Under analogous conditions, with the aid of formula (7), in § 2 of Chapter III there is proved (see Theorem 9) the validity of the formula

$$\Phi_k^{(i)}(x) = -\frac{1}{2\pi\sqrt{-1}}\sum_{\nu}\int_{C_\nu}\lambda^{m-1}\omega_k(x,\lambda,\Phi)\,d\lambda, \qquad (k = 0,\dots,q-1) \tag{8}$$

in the sense of the metric of $L_2(a_i, b_i)$, where C_ν is a simple closed contour enclosing only one pole λ_ν of the integrand, and the summation over ν extends over all poles; $\omega_k^{(i)}(x,\lambda,\Phi)$ $(i = 1,\dots,n)$ is a solution of the spectral problem

$$\omega_{k+1}^{(i)} - \lambda^m \omega_k^{(i)} = \Phi_k^{(i)}(x) \qquad (k = 0,\dots,q-2),$$

$$\sum_{\substack{mk+l\le p\\ k\le q-1}} \lambda^{mk} A_{kl}^{(i)}(x)\frac{d^l\omega_k^{(i)}}{dx^l} - \lambda^m\omega_{q-1}^{(i)} = \Phi_{q-1}^{(i)}(x) \tag{9}$$

for $x \in (a_i, b_i)$, $(i = 1, \dots, n)$,

$$\sum_{i=1}^{n} \sum_{\substack{l \le p-1 \\ k \le q}} \lambda^{mk} \left\{ \alpha_{kl}^{(i)} \left. \frac{d^l \omega_k^{(i)}}{dx^l} \right|_{x=a_i} + \beta_{kl}^{(i)} \left. \frac{d^l \omega_k^{(i)}}{dx^l} \right|_{x=b_i} \right\} = 0, \tag{10}$$

p, q, m are natural numbers, $p = mq$, $A_{kl}^{(i)}(x)$ is a square matrix of order r, and $\alpha_{kl}^{(i)}$, $\beta_{kl}^{(i)}$ are constant matrices of dimensions $nrp \times r$.

We note that (8) is the basic expansion formula. On the one hand, it expresses the fact of q-fold completeness of the system of eigenfunctions and associated functions (in the sense of M. V. Keldysh) of problem (9), (10); on the other hand, it makes it possible to compute the expansion coefficients.

From this formula, in particular, when $\Phi_k(x) = 0$ $(k = 0, \dots, q-2)$, one obtains a formula of type (3), generalizing the Langer-Birkhoff formula to the case of systems of equations with discontinuous coefficients, when the boundary conditions of the problem connect the derivatives of the sought vector-function at the points of discontinuity of the coefficients (see Theorem 10).

Chapter IV is devoted to the application of the author' s computational method to the solution, mainly, of two classes of one-dimensional mixed problems. In § 1 of Chapter IV the following mixed problem with nonseparating variables x and t is considered for a system of equations with discontinuous coefficients:

$$\frac{\partial^q u^{(i)}}{\partial t^q} = \sum_{\substack{mk+l \le p \\ k \le q}} A_{kl}^{(i)}\, x\, \frac{\partial^{k+l} u^{(i)}}{\partial t^k \partial x^l} + f^{(i)}(x, t) \tag{11}$$

for $x \in (a_i, b_i)$ $(i = 1, \dots, n)$,

$$\sum_{i=1}^{n} \sum_{\substack{1 \le p-1 \\ k \le q-1}} \left\{ \alpha_{kl}^{(i)} \left. \frac{\partial^{k+l} u^{(i)}}{\partial t^k \partial x^l} \right|_{x=a_i} + \beta_{kl}^{(i)} \left. \frac{\partial^{k+l} u^{(i)}}{\partial t^k \partial x^l} \right|_{x=b_i} \right\} = 0, \tag{12}$$

where

$$\left. \frac{\partial^k u^{(i)}}{\partial t^k} \right|_{t=0} = \Phi_k^{(i)}(x) \quad \text{for } x \in (a_i, b_i) \tag{13}$$

$$(k = 0, \dots, q-1), \quad (i = 1, \dots, n),$$

where $A_{kl}^{(i)}(x)$, $\alpha_{kl}^{(i)}$, $\beta_{kl}^{(i)}$ have the same meaning as in the corresponding spectral problem (9)–(10). Under the assumption that formula (8) is valid, a representation of a sufficiently smooth solution* of problem (11)–(13) is proved in the form of the complete residue of a certain meromorphic function determined by the solution of the corresponding spectral problem (see Theorem 11 on p. 171).

In § 2 of Chap. IV it is proved that if the boundary conditions (12) contain no derivatives with respect to time, then the residue representation of a sufficiently smooth solution of problem (11)–(13) can be obtained by means of an expansion formula of type (3)—Poincaré–Birkhoff (see Theorem 12 on p. 205).

In § 3 of Chap. IV a mixed problem is considered for a system of the form

$$M_1\left(t, \frac{\partial}{\partial t}\right) u^{(i)} = M_2\left(t, \frac{\partial}{\partial t}\right) L^{(i)}\left(x, \frac{\partial}{\partial x}\right) u^{(i)} + f^{(i)}(x,t). \tag{14}$$

Under a boundary condition not containing derivatives with respect to time, and with the aid of formula (8), a representation of a sufficiently smooth solution $u^{(i)}(x,t)$ is proved in the form of the complete residue of a definite meromorphic function (see Theorems 13, 14 on pp. 219, 223–224)

$$u^{(i)}(x,t) = -\frac{1}{2\pi\sqrt{-1}} \sum_{\nu} \int_{C_\nu} d\lambda \sum_{j=1}^{n} \int_{a_j}^{b_j} G^{(i,j)}(x,\xi,\lambda) y^{(j)}(t,\xi,\lambda)\, d\lambda, \tag{15}$$

where $G^{(i,j)}(x,\xi,\lambda)$ is the Green matrix of the corresponding spectral problem for the differential expression $L^{(i)}\left(x, \frac{\partial}{\partial x}\right) - \lambda$, and $y^{(i)}(t,\xi,\lambda)$ is the solution of the Cauchy problem

$$M_1\left(t, \frac{d}{dt}\right) y^{(i)} - \lambda M_2\left(t, \frac{d}{dt}\right) y^{(i)} = f^{(i)}(\xi,t), \tag{16}$$

$$\left.\frac{d^k y^{(i)}}{dt^k}\right|_{t=0} = \Phi_k^{(i)}(\xi). \tag{17}$$

It should be noted that from these assertions there follows directly the uniqueness of solutions of the mixed problems considered in Chap. IV in the class of sufficiently smooth functions in the above sense.

Chapter V is devoted to the application of the residue method to the solution of multidimensional problems.

In § 1 of Chap. V it is shown that, under the assumption of the basic expansion formula (8) for the corresponding spectral problem, an analogous residue formula also holds for a multidimensional mixed problem.

In § 2 of Chap. V a residue method of separation of variables is given, which is a generalization of the usual method of separation of variables to the case of nonorthogonal eigenfunctions. For the mixed problems with separable variables under consideration, the representability of sufficiently smooth solutions in the form of a multiple integral residue is proved (see Theorems 15, 16, 17 on pp. 232 –233, 234, 238).

* It is also assumed that the Green matrix commutes with the operators $M_i\left(t, \frac{\partial}{\partial t}\right)$.

Using the residue method for separation of variables given in § 2, § 3 of Chapter V proves the validity of formula (3) for a multidimensional spectral problem with separable variables, under the assumption that formula (3) is valid for each of the one-dimensional problems obtained by separation of variables (see Theorems 18, 19 on pp. 240, 242).

§ 4 of Chapter V is devoted to the application of the residue method of separation of variables to mixed problems of underground hydromechanics and the theory of heat conduction, solutions of which are absent from the literature.

The second part of the book (Chapters VI–X) is devoted to a systematic exposition of the author' s contour-integral method and its application to the investigation of one-dimensional and multidimensional mixed problems.

Chapters VI and VII are devoted to the application of this method to the investigation of one-dimensional mixed problems for equations of parabolic and hyperbolic types with discontinuous coefficients.

Chapters IX and X are devoted to the application of the contour-integral method to the investigation of multidimensional mixed problems for equations of parabolic type.

Unlike the residue method, the contour-integral method is also applicable to the solution of problems with continuous spectrum and makes it possible to prove the existence of solutions of the mixed problems under consideration, representable in the form of a contour integral. By this method the correctness of the mixed problems considered in the second part of the book is proved.

In comparison with the Laplace-transform method, it has the following advantage: it is also applicable to equations whose coefficients depend on time, and in the case of mixed problems for parabolic equations it is easily justified owing to the high rate of convergence of contour integrals.

Finally, by combining the residue method with the contour-integral method, one succeeds in indicating necessary and sufficient conditions for the solvability of the problems considered in the second part of the book.

In § 1 of Chapter VI a one-dimensional mixed problem is posed for second-order equations of parabolic type with discontinuous coefficients:

$$\frac{\partial v^{(i)}}{\partial t} = c_{02}^{(i)}(x)\frac{\partial^2 v^{(i)}}{\partial x^2} + c_{01}^{(i)}(x)\frac{\partial v^{(i)}}{\partial x} + c_{00}^{(i)}(x)v^{(i)} + f^{(i)}(x,t), \tag{18}$$

$$\sum_{i=1}^{n}\sum_{l=0}^{1}\sum_{k=0}^{1}\left\{\alpha_{slk}^{(i)}\frac{\partial^{\,l+k}v^{(i)}}{\partial t^k\partial x^l}\bigg|_{x=a_i}+\beta_{slk}^{(i)}\frac{\partial^{\,k+l}u^{(i)}}{\partial t^k\partial x^l}\bigg|_{x=b_i}\right\}=\gamma_s, \tag{19}$$

$$v^{(i)}(x,0)=\Phi^{(i)}(x),\qquad x\in(a_i,b_i),\qquad (i=1,\dots,2n), \tag{20}$$

where $\alpha_{slk}^{(i)}$, $\beta_{slk}^{(i)}$, γ_s are constants.

In § 2 of Chapter VI, under regularity conditions on the boundary conditions and a certain smoothness of the coefficients of equations (21) on the interval $[a_i,b_i]$, an asymptotic representation is given for the solution of the spectral problem

$$c_{02}^{(i)}(x)\frac{d^2y^{(i)}}{dx^2}+c_{01}^{(i)}(x)\frac{dy^{(i)}}{dx}+\left(c_{00}^{(i)}(x)-\lambda^2\right)y^{(i)}=\Phi^{(i)}(x) \tag{21}$$

for

$$x\in(a_i,b_i)\qquad (i=1,\dots,n),$$

$$\sum_{i=1}^{n}\sum_{l=0}^{1}\sum_{k=0}^{1}\left\{\alpha_{slk}^{(i)}\lambda^k\frac{d^ly^{(i)}}{dx^l}\bigg|_{x=a_i}+\beta_{slk}^{(i)}\lambda^k\frac{d^ly^{(i)}}{dx^l}\bigg|_{x=b_i}\right\}=\gamma_s\qquad (s=1,\dots,2n) \tag{22}$$

outside a certain δ-neighborhood of the spectrum, and it is assumed that the arguments of the ϑ-roots of the characteristic equation

$$c_{02}^{(i)}\vartheta^2-1=0 \tag{23}$$

do not depend on x (see Theorems 21, 22, 23 on pp. 265, 270 and 278). It should be noted that the obtained asymptotic representations of the solution of problem (21), (22) have not only auxiliary significance. They can also be used in solving the corresponding problems without initial conditions.

Under the conditions of § 2 of Chapter VI and conditions of sufficient smoothness of $f^{(i)}(x,t)$, $\Phi^{(i)}(x)$ for $x\in[a_i,b_i]$, in § 3 of Chapter VI it is proved that if

$$-\frac{\pi}{2}<\arg c_{02}^{(i)}(x)<\frac{\pi}{2}, \tag{24}$$

then problem (18)–(20) has a solution $v^{(i)}(x,t)$, representable in the form of a contour integral (see Theorems 24, 25 on pp. 281, 287).

In the same paragraph, under condition (24), the correctness of the mixed problem (18)–(20) is proved (see Theorems 26, 27, 28 on pp. 313, 316).

Moreover, in § 4 of Chapter VI it is proved that if

$$-\frac{\pi}{2} < \arg c_{02}^{(i)}(x) < \frac{3\pi}{2},$$

then problem (18)–(20), independently of the smoothness conditions on the data, cannot have a solution (see Theorem 29 on p. 319).

The fifth paragraph of Chapter VI is devoted to the application of the contour integral method to the study of one-dimensional mixed problems for second-order equations of Kovalevskaya type (the equations contain pure second derivatives with respect to x and with respect to t):

$$\frac{\partial^2 v^{(i)}}{\partial t^2} - \sum_{\substack{k+l\le 2\\ k\le 1}} c_{kl}^{(i)}(x)\,\frac{\partial^{k+l} v^{(i)}}{\partial t^k \partial x^l} = f^{(i)}(x,t), \qquad x \in (a_i, b_i) \quad (i = 1, \dots, n), \tag{25}$$

$$\sum_{i=1}^{n}\sum_{l=0}^{1}\left\{ \alpha_{sl}^{(i)}\left(\frac{\partial}{\partial t}\right) \frac{\partial^l v^{(i)}}{\partial x^l}\bigg|_{x=a_i} + \beta_{sl}^{(i)}\left(\frac{\partial}{\partial t}\right) \frac{\partial^l v^{(i)}}{\partial x^l}\bigg|_{x=b_i} \right\} = 0, \tag{26}$$

$$(s = 1, \dots, 2n),$$

$$v^{(i)}(x,0) = \Phi_0^{(i)}(x), \qquad \frac{\partial v^{(i)}}{\partial t}\bigg|_{t=0} = \Phi_1^{(i)}(x), \tag{27}$$

$$x \in (a_i, b_i) \quad (i = 1, \dots, n),$$

where $\alpha_{sl}^{(i)}(z)$, $\beta_{sl}^{(i)}(z)$ are polynomials in z with constant coefficients.

In this paragraph, first a more accurate asymptotic representation is obtained for the solution of the corresponding spectral problem outside a certain δ-neighborhood of the spectrum (see Theorem 30 on pp. 325–326).

With the aid of the obtained asymptotic representation of the solution of the spectral problem, under certain smoothness conditions on the data and regularity of the boundary conditions of the corresponding spectral problem it is proved that if the arguments of the θ-roots of the characteristic equations

$$c_{02}^{(i)}(x)\vartheta^2 + c_{11}^{(i)}(x)\vartheta - 1 = 0, \qquad x \in [a_i, b_i] \quad (i = 1, \dots, n) \tag{28}$$

do not depend on x and if the roots are real, then problem (25)–(27) has a unique solution, representable in the form of a contour integral with respect to a complex parameter.

When solving mixed problems for equations of hyperbolic type, in contrast to equations of parabolic type, closed contours are chosen.

Furthermore, it is proved that, under the conditions of Theorem 31, for the existence of a solution of the mixed problem (25)–(27), the reality of the θ-roots of the characteristic equations (28) is necessary, independently of the degree of smoothness of the data (see Theorem 32 on p. 341).

In § 1 of Chapter VII a suitable asymptotic representation is obtained for the solution of the spectral problem

$$\sum_{\substack{mk+l\le p\\ 0\le k\le q-1}} \lambda^{mk} A_{kl}^{(i)}(x)\frac{d^l y^{(i)}}{dx^l} - \lambda^{mq} y^{(i)} = F^{(i)}(x,\Phi,\lambda^m),$$

$$\sum_{i=1}^{n}\sum_{\substack{k\le q\\ l\le mq-1}} \lambda^{mk}\left\{\alpha_{slk}^{(i)}\frac{d^l y^{(i)}}{dx^l}\bigg|_{x=a_i} + \beta_{slk}^{(i)}\frac{d^l y^{(i)}}{dx^l}\bigg|_{x=b_i}\right\} = \gamma_s \tag{29}$$

outside the δ-neighborhood of the spectrum, where m is a natural number (see Theorems 33, 34 on pp. 355, 369).

In § 2 of Chapter VII the mixed problem

$$\frac{\partial u^{(i)}}{\partial t} = \sum_{l=0}^{p} A_{0l}^{(i)}(x)\frac{\partial^l u^{(i)}}{\partial x^l} + f^{(i)}(x,t), \tag{30}$$

$$\sum_{i=1}^{n}\sum_{\substack{k\leqslant l\\ l=p-1}}\left\{a_{skl}^{(i)}\,\frac{\partial^{k+l}u^{(i)}}{\partial t^k\partial x^l}\bigg|_{x=a_i} + \beta_{skl}^{(i)}\,\frac{\partial^{k+l}u^{(i)}}{\partial t^k\partial x^l}\bigg|_{x=b_i}\right\} = 0, \tag{31}$$

$$u^{(i)}(x,0) = \Phi^{(i)}(x) \quad \text{for } x\in(a_i,b_i) \quad (i=1,\dots,n). \tag{32}$$

It is proved that, under conditions of smoothness of the data of the problem and regularity of the boundary conditions (29) of the corresponding spectral problem (for $m=p,\ q=1$), if under the mapping

$$\lambda^m = z \tag{33}$$

all rays d_k $(k=1,\dots,2\mu\leqslant 2p)$ of the complex λ-plane, determined by the equations

$$\operatorname{Re}\lambda\varphi_j^{(i)}(x) = 0^*,$$

pass into rays lying in the left half-plane (and not coinciding with the imaginary semiaxes), then problem (30)–(32) has a unique solution representable in the form of a contour integral of a certain function of the complex parameter (see Theorems 35, 36 on pp. 372, 377).

Moreover, it has been proved that if, under the mapping (33), one of the rays d_j of the λ-plane passes into a ray lying in the right part of the z-plane (and not coinciding with the imaginary semiaxes), then problem (30)–(32), under the condition of regularity of the boundary conditions (29), irrespective of the degree of smoothness of the data, has no solution (see Theorem 37 on p. 378).

Chapters VIII and IX are devoted to the application of the contour integral method to the solution of the multidimensional mixed problem

$$c(x)M\left(t, \frac{\partial}{\partial t}\right)u = L\left(x, \frac{\partial}{\partial x}\right)u + f(x,t), \tag{34}$$

$$\lim_{x\to y} B\left(y, \frac{d}{dn_y}, M\right)u(x,t) = \psi(y)\exp\left(-\int_0^t b_0^{-1}(\tau)b_1(\tau)\,d\tau\right), \tag{35}$$

$$y \in \Gamma.$$

$$u(x,0) = \Phi(x), \qquad x \in D, \tag{36}$$

where

$$M = b_0(t)\frac{\partial}{\partial t} + b_1(t),$$

$$L\left(x, \frac{\partial}{\partial x}\right) = \sum_{i=1}^{3}\left(\frac{\partial^2}{\partial x_i^2} + a_i(x)\frac{\partial}{\partial x_i}\right) + a(x), \qquad x = (x_1, x_2, x_3),$$

$$B\left(y, \frac{\partial}{\partial y}, M\right) = \alpha_1(y)\frac{d}{dn_y} + \alpha_2(y)M\left(\frac{d}{dn_y} + \alpha_3(y)\right) + \alpha_4(y);$$

$c(x)$, $a_i(x)$ are continuously differentiable in the closure of the three-dimensional domain D with boundary Γ, which is a Lyapunov surface (D is bounded if the interior problem is posed, D is unbounded if the exterior problem is posed), $c(x) > 0$ for $x \in D + \Gamma$, n_y is the direction of the interior normal to Γ at the point y, ψ_y, $\alpha_i(y)$ are continuous on Γ, and $\alpha_1(y)$, $\alpha_2(y)$ do not vanish simultaneously on Γ; $b_k(t)$ are continuous for $t \in [0,T]$ and $b_0(t) > 0$.

To problem (34)–(36) there corresponds the following spectral problem and Cauchy problem with parameter

$$L\left(x, \frac{\partial}{\partial x}\right) v - \lambda^2 c(x) v = F(x), \tag{37}$$

$$\lim_{x \to y} B\left(y, \frac{d}{dn_y}, \lambda^2\right) v(x,\lambda) = \psi(y), \qquad y \in \Gamma, \tag{38}$$

$$M\left(t, \frac{d}{dt}\right) y - \lambda^2 y = f(\xi, t), \tag{39}$$

$$y(0) = \Phi(\xi) c(\xi). \tag{40}$$

Let R be a sufficiently large positive number, and let $\delta > 0$ be sufficiently small.

The following assertions are proved:

1. In the domain R_δ of values of λ satisfying the inequalities

$$|\lambda| \geq R, \qquad \cos(\arg\lambda) \geq \delta,$$

the solution of the spectral problem (37)–(38) exists and is an analytic function, decreasing well as $\lambda \in R_\delta$ increases (see Lemma 1, §3 and Lemma 3, §4, Chapter VIII).

2. Under the indicated restrictions on the data, the solution of the mixed problem (34)–(36) exists and is represented in the form of a contour integral

$$u(x,t) = \frac{1}{2\pi\sqrt{-1}} \int_S \left\{ \frac{y_1(t,\lambda)}{\lambda} v_1(x,\lambda) + \lambda \int_D G(x,\xi,\lambda) y(t,\xi,\lambda)\, d\xi \right\} d\lambda,$$

where S is an infinite contour situated in the domain R_δ, the branches of the contour S asymptotically approach the rays $\cos(\arg\lambda) = \delta$, $y_1(t,\lambda)$ is the solution of the homogeneous equation of problem (39), (40), satisfying the condition

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* $\varphi_j^{(i)}(x)$ are the ϑ-roots of the characteristic equations

$$A_{0p}^{(i)}(x)\vartheta^p - 1 = 0.$$

$y_1(0, \lambda) = 1$, $v_1(x, \lambda)$ is the solution of problem (37), (38) for the homogeneous equation, $G(x, \xi, \lambda)$ is the Green’ s function of problem (37), (38), and $y(t, \xi, \lambda)$ is the solution of problem (39), (40). (See Theorem 38, §1 and Theorem 39, §2, Chapter IX.)

By the same method, in Chapter X analogous results are obtained for the mixed problem of the form (34)–(36), for an equation with discontinuous coefficients (see Lemmas 1, §2, 2, §3 and Theorems 40, 41, §4, Chapter X).

M. L. Rasulova’ s book, in addition to a systematic exposition of the residue method and the contour integral method, also contains a number of new and interesting results in the spectral theory of linear differential operators.

One of the achievements of the book is that it not only develops the investigations of Koshlyakov, Puankare, Birkhoff, Tamarkin, and other authors concerned with the spectral theory of differential operators, but also provides new effective methods for solving broad classes of mixed problems containing time derivatives in the boundary conditions, problems not previously considered in the literature.

The book is a valuable contribution to the theory of differential equations with partial derivatives and a useful aid on equations of mathematical physics.

It seems to us that many of the results contained in M. L. Rasulova’ s book will be useful not only in the theoretical sense, but will also be applied to the solution of particular practical problems.

Doctor of Physical and Mathematical Sciences **V. I. KRYLOV**

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.