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Abstract

Full Text

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ON THE FACTORIZATION OF FINITE GROUPS BY MEANS OF INDEXIALS

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§ 1.

In the paper ([1]) we proposed a general method of factorization of finite groups, based on the use of the orders of the factor groups of a principal series of the group.

The question of the possibility of using, for the same purpose, "smaller" elements —the orders of subgroups of the indicated factor groups—has apparently not been raised until now. Indirectly, one such case was indicated by us in ([2]).

Theorems 1 and 2 of the present paper give a positive answer to this question in one further case. At the same time, these theorems are parallel also to the criterion for factorization of a group into two factors contained in Theorem 1 on groups of type Π-2 from ([2]), generalizing it in the particular case in which the normal series considered there is also invariant.

As is not difficult to see from a comparison of the definitions of groups of type Π-2 (see ([2])) and of the indexial of a group used below, this generalization is achieved by dropping, in the hypotheses of the theorem on groups of type Π-2, the coprimeness of the orders of the factors and the solvability of one of them, while the requirement that certain subgroups be conjugate is weakened.

Theorems 3 and 4 given below generalize Theorem E1* from ([3]). Thus, in the present paper we continue the study begun by us in ([2]) of the connections between the factorization properties of a group and those of the factor groups of its normal and invariant series. In doing so we use the notation and results of the method of indexials set out in ([4]).

§ 2.

Let

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \cdots \supset \mathfrak{G}_\lambda = \mathfrak{E} \qquad \text{(R)}$$

be some series of normal divisors of the finite group $\mathfrak{G} \neq \mathfrak{E}$ (i.e. its invariant series). In the case $\mathfrak{G} = \mathfrak{E}$, as the series (R) we shall consider the series $\mathfrak{E}, \mathfrak{E}, \dots, \mathfrak{E}$

($\lambda + 1$ times).

Suppose that, for the factor groups of the series (R), the factorizations

$$\left.\begin{array}{c}\mathfrak{G}_{i-1}/\mathfrak{G}_i = [\mathfrak{F}_{i,1}/\mathfrak{G}_i]\,[\mathfrak{F}_{i,2}/\mathfrak{G}_i], \quad i = 1, 2, \dots, \lambda, \\ \text{for which the subgroups} \\ \mathfrak{F}_{1,j}/\mathfrak{G}_1,\ \mathfrak{F}_{2,j}/\mathfrak{G}_2,\ \dots,\ \mathfrak{F}_{\lambda,j}/\mathfrak{G}_\lambda, \quad j = 1, 2, \\ \text{are such that one can form (see (}^4\text{)) the indexial} \\ (\mathfrak{F}_{1,j}/\mathfrak{G}_1)(\mathfrak{F}_{2,j}/\mathfrak{G}_2)\cdots(\mathfrak{F}_{\lambda,j}/\mathfrak{G}_\lambda) = (h_j)_R. \end{array}\right\} \tag{F}$$

are valid.

Definition 1. The system of factorizations (F) will be called an **indexial system** of the series (R) of the group $\mathfrak{G}$, or, more briefly, an indexial system of the group $\mathfrak{G}$.

Let us note that in the factorizations (F) we impose no restrictions on the orders of the factors. Taking one of them to be the identity subgroup $\mathfrak{G}_i/\mathfrak{G}_i$, we see that every invariant series of any finite group has at least one indexial system.

By Theorem 6 of [4], the indexial $(h_j)_R,\ j = 1, 2$, has at least one regular Sylow expansion

$$(\mathfrak{F}_{1,j}/\mathfrak{G}_1)(\mathfrak{C}_{2,j}/\mathfrak{G}_2)\cdots(\mathfrak{C}_{\lambda,j}/\mathfrak{G}_\lambda) = (c_j h_j)_R, \qquad j = 1, 2. \tag{1}$$

Then, on the basis of the definition of a regular indexial [4], there exists at least one suitable subgroup for the indexial (1). Let $\mathfrak{H}_j,\ j = 1, 2$, be any one of them. By the property of a regular Sylow expansion of an indexial (see [4]), we have $(\mathfrak{H}_j) = c_j h_j$ and $\Pi(c_j) \subset \Pi(h_j),\ j = 1, 2$. The subgroups $\mathfrak{H}_j$ have a number of other properties as well, a list of which can be found in [4].

Definition 2. A pair of subgroups $\mathfrak{H}_1, \mathfrak{H}_2$ will be called a **suitable pair of subgroups of the indexial system** (F) of the series (R) of the group $\mathfrak{G}$.

It follows immediately from the preceding that

Theorem 1. *For every indexial system of a group $\mathfrak{G}$ there exists at least one pair of suitable subgroups.*

Theorem 2. *If $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are some subgroups of a group $\mathfrak{G}$, then for the existence of a factorization $\mathfrak{G} = \mathfrak{H}_1\mathfrak{H}_2$ it is necessary and sufficient that the pair of subgroups $\mathfrak{H}_1, \mathfrak{H}_2$ be a suitable pair of subgroups for some indexial system of the group $\mathfrak{G}$.*

Proof. Sufficiency. Suppose that the theorem is false. Then let $\mathfrak{G}$ be one of the groups of least order for which the theorem does not hold. This means that there exists such a pair of suitable subgroups $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of some indexial system (F) of the group $\mathfrak{G}$ that $\mathfrak{G} \neq \mathfrak{H}_1\mathfrak{H}_2$. Since, for $\mathfrak{G} = \mathfrak{E}$, we would have $\mathfrak{H}_1 = \mathfrak{E}$, $\mathfrak{H}_2 = \mathfrak{E}$, and $\mathfrak{G} = \mathfrak{H}_1\mathfrak{H}_2$, it follows that $\mathfrak{G} \neq \mathfrak{E}$.

The subgroups $\mathfrak{H}_1$ and $\mathfrak{H}_2$ correspond to the indexials $(h_1)_R$ and $(h_2)_R$ of the series (R). Consider the factorization

$$\mathfrak{G}/\mathfrak{G}_1 = [\mathfrak{F}_{1,1}/\mathfrak{G}_1]\,[\mathfrak{F}_{1,2}/\mathfrak{G}_1]. \tag{2}$$

Let first $\lambda = 1$, i.e., let $\mathfrak{G}_1 = \mathfrak{E}$. Then $\mathfrak{G} = \mathfrak{F}_{1,1}\mathfrak{F}_{1,2}$.

According to the definition of a suitable subgroup of an indexial and in view of $\mathfrak{G}_1 = \mathfrak{E}$, we have

$$\mathfrak{F}_{1,1} = [\mathfrak{H}_1 \cap \mathfrak{G}_0]\mathfrak{G}_1 = \mathfrak{H}_1, \qquad \mathfrak{F}_{1,2} = [\mathfrak{H}_2 \cap \mathfrak{G}_0]\mathfrak{G}_1 = \mathfrak{H}_2.$$

Thus, $\mathfrak{G} = \mathfrak{H}_1\mathfrak{H}_2$. We have obtained a contradiction.

Hence, $\lambda > 1$, i.e., $\mathfrak{G}_2$ exists. It is now not difficult to verify that, since (1) is an indexial of the series (R), the product $(\mathfrak{C}_{2,j}/\mathfrak{G}_2)(\mathfrak{C}_{3,j}/\mathfrak{G}_3)\cdots(\mathfrak{C}_{\lambda,j}/\mathfrak{G}_\lambda)$, $j = 1, 2$, will be an indexial for the series

$$\mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \cdots \supset \mathfrak{G}_\lambda = \mathfrak{E}. \tag{R}$$

We shall now show that

$$\left.\begin{array}{c} \mathfrak{H}_j \cap \mathfrak{G}_1 = \mathfrak{H}'_j, \qquad j = 1, 2, \\ \text{will be a suitable subgroup for the indexial} \\ (\mathfrak{G}_1/\mathfrak{G}_1)(\mathfrak{C}_{2,j}/\mathfrak{G}_2)\cdots(\mathfrak{C}_{\lambda,j}/\mathfrak{G}_\lambda) \\ \text{of the series (R) and for the indexial} \\ (\mathfrak{C}_{2,j}/\mathfrak{G}_2)\cdots(\mathfrak{C}_{\lambda,j}/\mathfrak{G}_\lambda) \\ \text{of the series } (R_1). \end{array}\right\} \tag{3}$$

Since $\mathfrak{h}'_j \subseteq \mathfrak{G}_1 \subset \mathfrak{G}_0 = \mathfrak{G}$, we have

$$\mathfrak{G}_1 = [\mathfrak{h}'_j \cap \mathfrak{G}_0]\mathfrak{G}_1. \tag{4}$$

Further, taking into account that $\mathfrak{h}_j$, $j = 1, 2$, is a suitable subgroup for the indexial (1) and that $\mathfrak{G}_{i-1} \supset \mathfrak{G}_i$, $i = 2, 3, \dots, \lambda$, we have:

$$\mathfrak{G}_{i,j} = [\mathfrak{h}_j \cap \mathfrak{G}_{i-1}]\mathfrak{G}_i = [\mathfrak{h}'_j \cap \mathfrak{G}_{i-1}]\mathfrak{G}_i. \tag{5}$$

But (4), (5), and $\mathfrak{h}'_j \subseteq \mathfrak{G}_1$, on the basis of the definition of a suitable subgroup (see (4)), prove (3).

We shall further distinguish the following two cases.

1) $\mathfrak{G}/\mathfrak{G}_1 = \mathfrak{F}_{1,1}/\mathfrak{G}_1$. In this case it is obvious that $\mathfrak{F}_{1,1} = \mathfrak{G} = \mathfrak{h}_1\mathfrak{G}_1$ and that from (F), on the basis of the definition of an indexial (4), for $i = 2, 3, \dots, \lambda$ there is obtained an indexial system of the series (R_1) of the group $\mathfrak{G}_1$. Replacing in (F) the factors $\mathfrak{F}_{i,j}/\mathfrak{G}_i$ by their extensions $\mathfrak{C}_{i,j}/\mathfrak{G}_i$, we obtain the new factorization

$$\mathfrak{G}_{i-1}/\mathfrak{G}_i = [\mathfrak{C}_{i,1}/\mathfrak{G}_i]\,[\mathfrak{C}_{i,2}/\mathfrak{G}_i], \quad i = 2, 3, \dots, \lambda. \tag{F'}$$

It is obvious that the equalities (F′) are also an indexial system of the series (R_1) of the group $\mathfrak{G}_1$.

Since the series (R) has no repetitions, $(\mathfrak{G}_1) < (\mathfrak{G})$, and therefore the theorem will be true for $\mathfrak{G}_1$. From this it follows, taking (3) into account, that the following factorization of the group $\mathfrak{G}_1$ holds:

$$\mathfrak{G}_1 = \mathfrak{h}_1\mathfrak{h}_2' = [\mathfrak{h}_1 \cap \mathfrak{G}_1]\,[\mathfrak{h}_2 \cap \mathfrak{G}_1]. \tag{6}$$

As we established above, $\mathfrak{G} = \mathfrak{h}_1\mathfrak{G}_1$. Hence, in view of (6), we have

$$\mathfrak{G} = \mathfrak{h}_1[\mathfrak{h}_1 \cap \mathfrak{G}_1][\mathfrak{h}_2 \cap \mathfrak{G}_1] = \mathfrak{h}_1[\mathfrak{h}_2 \cap \mathfrak{G}_1] \subseteq \mathfrak{h}_1\mathfrak{h}_2.$$

But $\mathfrak{h}_1\mathfrak{h}_2 \subseteq \mathfrak{G}$. Consequently, $\mathfrak{G} = \mathfrak{h}_1\mathfrak{h}_2$—a contradiction.

2) $\mathfrak{G}/\mathfrak{G}_1 \neq \mathfrak{F}_{1,1}/\mathfrak{G}_1$. Consider the proper subgroup $\mathfrak{F}_{1,1}/\mathfrak{G}_1$ of the group $\mathfrak{G}/\mathfrak{G}_1$. If it were the case that $\mathfrak{F}_{1,1} = \mathfrak{G}_1$, then, obviously, $\mathfrak{G}/\mathfrak{G}_1 = [\mathfrak{G}_1/\mathfrak{G}_1] \times [\mathfrak{F}_{1,2}/\mathfrak{G}_1] = \mathfrak{F}_{1,2}/\mathfrak{G}_1$, and we would again arrive at a contradiction, arguing as in case 1). Hence, taking into account the inequality $\lambda > 1$ established earlier, we conclude that there exists a series without repetitions

$$\mathfrak{F}_{1,1} \supset \mathfrak{G}_1 \supset \mathfrak{G}_2 \supset \cdots \supset \mathfrak{G}_\lambda = \mathfrak{E}, \tag{R_2}$$

and moreover the factorizations

$$\mathfrak{F}_{1,1}/\mathfrak{G}_1 = [\mathfrak{F}_{1,1}/\mathfrak{G}_1]\,[\mathfrak{G}_1/\mathfrak{G}_1],$$

$$\mathfrak{G}_{i-1}/\mathfrak{G}_i = [\mathfrak{F}_{i,1}/\mathfrak{G}_i]\,[\mathfrak{F}_{i,2}/\mathfrak{G}_i], \quad i = 2, 3, \dots, \lambda. \tag{7}$$

Replacing in the last of the equalities (7) the groups $\mathfrak{F}_{i,j}/\mathfrak{G}_i$ by the extensions $\mathfrak{C}_{i,j}/\mathfrak{G}_i$, we arrive at the following factorizations

$$\mathfrak{F}_{1,1}/\mathfrak{G}_1 = [\mathfrak{F}_{1,1}/\mathfrak{G}]\,[\mathfrak{G}_1/\mathfrak{G}_1],$$

$$\mathfrak{G}_{i-1}/\mathfrak{G}_i = [\mathfrak{C}_{i,1}/\mathfrak{G}_i]\,[\mathfrak{C}_{i,2}/\mathfrak{G}_i], \quad i = 2, 3, \dots, \lambda. \tag{F''}$$

According to the definition of a suitable subgroup, we have: $\mathfrak{F}_{1,1} = \mathfrak{h}_1\mathfrak{G}_1$ and $\mathfrak{F}_{1,2} = \mathfrak{h}_2\mathfrak{G}_1$. Therefore $\mathfrak{G}/\mathfrak{G}_1 = [\mathfrak{F}_{1,1}/\mathfrak{G}_1]\,[\mathfrak{F}_{1,2}/\mathfrak{G}_1] = [\mathfrak{h}_1\mathfrak{G}_1/\mathfrak{G}_1]\,[\mathfrak{h}_2\mathfrak{G}_1/\mathfrak{G}_1]$. But then

$$\mathfrak{G} = \mathfrak{h}_1\mathfrak{h}_2\mathfrak{G}_1 = \mathfrak{F}_{1,1}\mathfrak{h}_2. \tag{8}$$

It is not hard to verify that $(\mathfrak{F}_{1,1}/\mathfrak{G}_1)(\mathfrak{C}_{2,1}/\mathfrak{G}_2)\dots(\mathfrak{C}_{\lambda,1}/\mathfrak{G}_\lambda)$ and $(\mathfrak{G}_1/\mathfrak{G}_1) \times (\mathfrak{C}_{2,2}/\mathfrak{G}_2)\dots(\mathfrak{C}_{\lambda,2}/\mathfrak{G}_\lambda)$ will be indexials also for the group $\mathfrak{F}_{1,1}$ with respect to the series (R_2). This means that the equalities (F'') form an indexial system of the series (R_2) of the group $\mathfrak{F}_{1,1}$.

By hypothesis, $(\mathfrak{F}_{1,1}) < (\mathfrak{G})$. Therefore the theorem will be valid for $\mathfrak{F}_{1,1}$. Hence, taking into consideration that from $\mathfrak{F}_{1,1} = \mathfrak{h}_1\mathfrak{G}_1$ it follows that $\mathfrak{h}_1 \subseteq \mathfrak{F}_{1,1}$, and also that the replacement of (R) by (R_2) preserves (3), we see,

that for $\mathfrak{F}_{1,1}$ the factorization $\mathfrak{F}_{1,1} = \mathfrak{H}_1[\mathfrak{H}_2 \cap \mathfrak{G}_1]$ holds. Hence, on the basis of (8), we obtain:

$$\mathfrak{G} = \mathfrak{F}_{1,1}\mathfrak{H}_2 = \mathfrak{H}_1[\mathfrak{H}_2 \cap \mathfrak{G}_1]\mathfrak{H}_2 = \mathfrak{H}_1\mathfrak{H}_2.$$

Again we have obtained a contradiction.

Necessity. Let $\mathfrak{G} = \mathfrak{H}_1\mathfrak{H}_2$, where $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are some subgroups of $\mathfrak{G}$. Then $\mathfrak{H}_1$ and $\mathfrak{H}_2$ will, obviously, be suitable subgroups of the indexials $(\mathfrak{H}_1/\mathfrak{G})$ and $(\mathfrak{H}_2/\mathfrak{G})$ of the series $\mathfrak{G} \supseteq \mathfrak{E}$ of the group $\mathfrak{G}$.

§ 3. **Theorem 3.** *If $\mathfrak{G}$ has a normal divisor $\mathfrak{G}_1$, and a factorization*

$$\mathfrak{G}/\mathfrak{G}_1 = [\mathfrak{F}_1/\mathfrak{G}_1][\mathfrak{F}_2/\mathfrak{G}_1]\cdots[\mathfrak{F}_k/\mathfrak{G}_1]$$

holds, then the factorization $\mathfrak{G} = \overline{\mathfrak{F}}_1\overline{\mathfrak{F}}_2\cdots\overline{\mathfrak{F}}_k\mathfrak{G}_1$ also holds, where $\overline{\mathfrak{F}}_i$ is a subgroup of $\mathfrak{G}$ for which $\overline{\mathfrak{F}}_i \cap \mathfrak{G}_1 = \mathfrak{C}_i$ is a special subgroup, and moreover $\overline{\mathfrak{F}}_i/\mathfrak{C}_i \simeq \mathfrak{F}_i/\mathfrak{G}_1$ and $\Pi((\mathfrak{C}_i)) \subseteq \Pi((\mathfrak{F}_i/\mathfrak{G}_1))$, $i = 1, 2, \dots, k$.

Proof. Consider the indexial $(\mathfrak{F}_i/\mathfrak{G}_1)(\mathfrak{G}/\mathfrak{G})$, $i = 1, 2, \dots, k$, of the series $\mathfrak{G} \supseteq \mathfrak{G}_1 \supseteq \mathfrak{E}$. By Theorem 6 ([4]), this indexial has a regular s.a.z. extension $(\mathfrak{F}_i/\mathfrak{G}_1)(\mathfrak{C}_i/\mathfrak{G})$, an arbitrary suitable subgroup of which we denote by $\overline{\mathfrak{F}}_i$. By the definition of a suitable subgroup ([4]), we have:

$$\mathfrak{F}_i = \overline{\mathfrak{F}}_i\mathfrak{G}_1, \qquad \mathfrak{C}_i = [\overline{\mathfrak{F}}_i \cap \mathfrak{G}_1]\mathfrak{E} = \overline{\mathfrak{F}}_i \cap \mathfrak{G}_1. \tag{9}$$

Hence $\overline{\mathfrak{F}}_i/\mathfrak{C}_i \simeq \mathfrak{F}_i\mathfrak{G}_1$. According to the definition of a s.a.z. extension of an indexial ([4]), the subgroup $\mathfrak{C}_i$ is special and $\Pi((\mathfrak{C}_i)) \subseteq \Pi((\mathfrak{F}_i/\mathfrak{G}_1))$.

Further, from the conditions of the theorem and (9) it follows that

$$\mathfrak{G} = \mathfrak{F}_1\mathfrak{F}_2\cdots\mathfrak{F}_k = \overline{\mathfrak{F}}_1\overline{\mathfrak{F}}_2\cdots\overline{\mathfrak{F}}_k\mathfrak{G}_1.$$

All these properties of the subgroups $\overline{\mathfrak{F}}_1, \overline{\mathfrak{F}}_2, \dots, \overline{\mathfrak{F}}_k$ show that the theorem holds for $\mathfrak{G}$.

Theorem 4. *If $\mathfrak{G}$ has a normal series $\mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \cdots \supseteq \mathfrak{G}_\lambda = \mathfrak{E}$, and factorizations*

$$\mathfrak{G}_{i-1}/\mathfrak{G}_i = [\mathfrak{F}_{i,1}/\mathfrak{G}_i] \times [\mathfrak{F}_{i,2}/\mathfrak{G}_i]\cdots[\mathfrak{F}_{i,k_i}/\mathfrak{G}_i], \quad i = 1, 2, \dots, \lambda, \quad k_i \geq 1,$$

hold, then the factorization

$$\mathfrak{G} = [\overline{\mathfrak{F}}_{1,1}\overline{\mathfrak{F}}_{1,2}\cdots\overline{\mathfrak{F}}_{1,k_1}][\overline{\mathfrak{F}}_{2,1}\overline{\mathfrak{F}}_{2,2}\cdots\overline{\mathfrak{F}}_{2,k_2}]\cdots[\overline{\mathfrak{F}}_{\lambda,1}\overline{\mathfrak{F}}_{\lambda,2}\cdots\overline{\mathfrak{F}}_{\lambda,k_\lambda}]$$

also holds, where $\overline{\mathfrak{F}}_{i,j}$ is a subgroup of $\mathfrak{G}$ for which $\overline{\mathfrak{F}}_{i,j} \cap \mathfrak{G}_i = \mathfrak{C}_{i,j}$ is a special subgroup, and moreover $\overline{\mathfrak{F}}_{i,j}/\mathfrak{C}_{i,j} \simeq \mathfrak{F}_{i,j}/\mathfrak{G}_i$ and $\Pi((\mathfrak{C}_{i,j})) \subseteq \Pi((\mathfrak{F}_{i,j}/\mathfrak{G}_i))$, and $\overline{\mathfrak{F}}_{i,j} \cap \overline{\mathfrak{F}}_{i_1,j_1}$ is a special group when $i < i_1$.

Proof. Obviously, it is enough to apply Theorem 3 successively to the pairs of groups $\mathfrak{G}$ and $\mathfrak{G}_1$, $\mathfrak{G}_1$ and $\mathfrak{G}_2, \dots, \mathfrak{G}_{\lambda-1}$ and $\mathfrak{G}_\lambda$.

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Note: Figure translations are in progress. See original paper for figures.