ON INTEGRAL OPERATORS WITH CARLEMAN KERNELS

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V. B. KOROTKOV

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Let Ω be a measurable subset of n-dimensional Euclidean space R_n. A **Carleman kernel** is a complex-valued function $K(s,t)$ defined on $\Omega \times \Omega$ and satisfying three conditions:

A. $K(s,t)$ is measurable.

B. For almost all s in Ω,

$$\int_\Omega |K(s,t)|^2 \, dt < \infty.$$

C. For almost all (s,t),

$$K(s,t) = \overline{K(t,s)}.$$

A linear (in general, unbounded) operator T, defined on an everywhere dense linear manifold D_T in $L_2(\Omega)$, is called a **Carleman operator** if it is representable in the form

$$(Tf)(s) = \int_\Omega K(s,t) f(t) \, dt, \qquad f \in D_T, \tag{1}$$

where $K(s,t)$ is a Carleman kernel.

Operators of this kind were first considered by T. Carleman ([1]). The problem of describing the specific properties of Carleman operators was posed by J. von Neumann in the paper ([2]), where there are two distinct formulations of the problem. In the first, the question concerns the characteristic properties of the spectrum of Carleman operators; in the second, the question concerns finding conditions for representability of an operator in the form (1) with a Carleman kernel. In his paper J. von Neumann considered the problem in the first formulation and, for self-adjoint operators, obtained a definitive result.

In the second formulation the problem was studied by N. I. Akhiezer ([3]). In this work N. I. Akhiezer singled out from the totality of all Carleman kernels a certain class of kernels and found a criterion for representability of a symmetric operator in the form (1) with a kernel from this class.

In the present paper a solution of the problem is given for the entire totality of Carleman kernels. The criterion is formulated in Theorem 2. Throughout the paper the linear operator T is assumed to be defined on an everywhere dense linear manifold D_T in $L_2(\Omega)$.

1. **Operators of Carleman type.** We shall call a linear operator T an **integral operator of Carleman type** if it is representable in the form (1) with a kernel satisfying only conditions A and B.

Definition. We shall say that a **linear operator** T **has a majorant** if there exists a measurable, almost everywhere finite, nonnegative function $\Lambda(s)$ such that for any function $f \in D_T$, $\|f\| = 1$, almost everywhere

$$|(Tf)(s)| \leq \Lambda(s). \tag{2}$$

The function $\Lambda(s)$ will be called a **majorant**.

Lemma 1. Let $\varphi(s)$, $s \in \Omega$, be a measurable abstract function taking values in $L_2(\Omega)$. Then there exists a kernel $l(s,t)$, satisfying conditions A and B, such that for almost all $s \in \Omega$, $l(s,t)$, as a function of t, coincides with $\varphi(s)$.

Theorem 1. *A linear operator T is an integral operator of Carleman type if and only if T has a majorant.*

Necessity. As a majorant one may choose the function

$$L(s) = \left(\int_\Omega |K(s,t)|^2 dt \right)^{1/2}.$$

Sufficiency. Let $\Lambda(s)$ be a majorant of the operator T. Put

$$E_k = E(s:\ \Lambda(s) \leq k), \qquad V_k = E(s:\ s \in R_n,\ \|s\| \leq k)$$

and

$$\Omega_k = \Omega \cap E_k \cap V_k, \qquad k = 1, 2, \dots .$$

Then $\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3 \subseteq \dots$ and

$$m\left(\Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k\right) = m\left(\Omega \setminus \bigcup_{k=1}^{\infty} E_k\right) = 0. \tag{3}$$

Define, on the linear set D_T, everywhere dense in $L_2(\Omega)$, the functionals

$$F_E^{(k)}(f) = \int_E (Tf)(s)\,ds, \qquad E \subseteq \Omega_k, \qquad k = 1, 2, \dots . \tag{4}$$

By virtue of (2), for any $f \in D_T$, $\|f\| = 1$,

$$|F_E^{(k)}(f)| \leq \int_{E \cap E_k} |(Tf)(s)|\,ds \leq k mE, \qquad E \subseteq \Omega_k.$$

Consequently, the linear and bounded functionals $F_E^{(k)}$ on D_T can be extended by continuity to the whole space $L_2(\Omega)$, and moreover

$$\|F_E^{(k)}\| \leq kmE, \qquad E \subseteq \Omega_k.$$

But then, by the theorem of N. Dunford and B. Pettis ((4), p. 584), there exists, and moreover uniquely (up to equivalence), a measurable abstract function $\varphi_k(s)$, $s \in \Omega_k$, such that

$$F_E^{(k)} = \int_E \varphi_k(s)\, ds, \qquad E \subseteq \Omega_k. \tag{5}$$

Extend $\varphi_k(s)$ by zero for $s \in \Omega \setminus \Omega_k$. By (4) and by the uniqueness, $\varphi_k(s) = \varphi_{k+1}(s)$ for almost all $s \in \Omega_k$, $k = 1, 2, \ldots$. Hence, for almost all $s \in \Omega$, $\lim_{k\to\infty} \varphi_k(s)$ exists; moreover the limiting function $\varphi(s)$ is also measurable and $\varphi(s) = \varphi_k(s)$ for almost all $s \in \Omega_k$, $k = 1, 2, \ldots$. Let $f \in D_T$. Since for every measurable set E from Ω_k, $k = 1, 2, \ldots$,

$$\int_E (Tf)(s)\, ds = F_E^{(k)}(f) = \left\langle f, \int_E \varphi_k(s)\, ds \right\rangle = \int_E (f, \varphi_k(s))\, ds = \int_E (f, \varphi(s))\, ds,$$

it follows, by (3), that for almost all $s \in \Omega$,

$$(Tf)(s) = (f, \varphi(s)).$$

The proof is completed by applying Lemma 1.

Lemma 2. *Let T be an integral operator of Carleman type with kernel $K(s,t)$. Then every majorant of it, for almost all $s \in \Omega$, satisfies the inequality*

$$\left(\int_\Omega |K(s,t)|^2\, dt \right)^{1/2} \leq \Lambda(s). \tag{6}$$

Proof. Let F, $F \subset D_T$, be a countable set lying on the unit sphere of $L_2(\Omega)$ and everywhere dense in it. Using the definition of a majorant and the countability of F, we find in Ω a set N_0 of measure zero such that for all $s \in \Omega \setminus N_0$ and $f \in F$,

$$\left| \int_\Omega K(s,t) f(t)\, dt \right| \leq \Lambda(s).$$

From this (6) follows immediately.

II. Carleman Operators

Let $\xi(s) \geq 0$, $s \in \Omega$, be a measurable function finite almost everywhere. It is known that the linear manifold

$$[L_2(\Omega)]_\xi = (f: \ f \in L_2(\Omega), \ \int_\Omega \xi(s)|f(s)|\,ds < \infty)$$

is everywhere dense in $L_2(\Omega)$ ([3]).

Theorem 2. *In order that a linear operator T be a Carleman integral operator, it is necessary and sufficient that:*

1) *the operator T have a majorant $\Lambda(s)$;*

2) $D_{T^*} \supseteq [L_2(\Omega)]_\Lambda$*;*

3) *for any f, g from $[L_2]_\Lambda$, $(T^*f, g) = (f, T^*g)$.*

Necessity. Put

$$K(s) = \left(\int_\Omega |K(s,t)|^2 dt\right)^{1/2}.$$

Following (3), consider the linear manifold D_B of all functions $f \in L_2(\Omega)$ for which

$$\int_\Omega K(s,t)f(t)\,dt = \varphi_f(s) \in L_2(\Omega)$$

and define on the linear manifolds D_B and $[L_2]_K \subseteq D_B$ linear operators B and A:

$$Bf = \varphi_f, \quad f \in D_B, \qquad Ag = \varphi_g, \quad g \in [L_2]_K.$$

A is a symmetric operator and $A^* = B$ ([3]). Let T be a Carleman integral operator with kernel $K(s,t)$. By Theorem 1, T has a majorant $\Lambda(s)$. Since $T \subseteq B$, we have $T^* \supseteq B^* = A^{**} \supseteq A$. Consequently,

$$D_{T^*} \supseteq D_A = [L_2]_K \supseteq \text{(see (6))} \supseteq [L_2]_\Lambda.$$

The third assertion follows from the inclusions $T^* \supseteq A$, $D_A \supseteq [L_2]_\Lambda$ and the symmetry of the operator A.

Sufficiency. Since T has a majorant $\Lambda(s)$, by Theorem 1, T is representable in the form (1) with a kernel $K(s,t)$ satisfying conditions A, B. It remains to show that the kernel $K(s,t)$ is Hermitian. As above, construct from the kernel $K(s,t)$ an operator B. It is clear that $T \subseteq B$. Since for any $f \in [L_2]_\Lambda$ and $g \in L_2(\Omega)$,

$$\int_\Omega \int_\Omega |K(s,t)|\,|f(s)|\,|g(t)|\,dt\,ds \le \|g\| \int_\Omega K(s)|f(s)|\,ds \le \|g\| \int_\Omega \Lambda(s)|f(s)|\,ds, \tag{7}$$

we have

$$\int_\Omega \overline{K(s,t)}\,f(s)\,ds \in L_2(\Omega).$$

This makes it possible to define on $[L_2]_\Lambda$ the linear operator A:

$$(Af)(t) = \int_\Omega \overline{K(s,t)}\,f(s)\,ds, \qquad f \in [L_2(\Omega)]_\Lambda.$$

In view of (7) and Fubini' s theorem, for any $g \in D_B$ and $f \in D_A = [L_2]_\Lambda$,

$$\int_\Omega \left(\int_\Omega \overline{K(s,t)}\,f(s)\,ds \right) \overline{g(t)}\,dt = \int_\Omega f(s) \left(\int_\Omega K(s,t)\overline{g(t)}\,dt \right) ds.$$

Consequently, $B \subseteq A^*$. But $T \subseteq B$. Therefore $A \subseteq A^{**} \subseteq T^*$. Hence, by condition 3, for any f and g from $D_A = [L_2]_\Lambda$,

$$\int_\Omega \left(\int_\Omega \overline{K(s,t)}\,f(s)\,ds \right) \overline{g(t)}\,dt = \int_\Omega f(s) \overline{\left(\int_\Omega \overline{K(t,s)}\,g(t)\,dt \right)} ds.$$

Interchanging (on the basis of (7)) the order of integration on the left and using the fact that f and g range over the everywhere dense set $[L_2(\Omega)]_\Lambda$ in $L_2(\Omega)$, we obtain that $\overline{K(s,t)} = \overline{K(t,s)}$ for almost all $t \in \Omega$ for almost every $s \in \Omega$. The theorem is proved.

III. Operators of Special Form

Let the linear operator T have the form

$$Tf = \sum_{k=1}^{\infty} \lambda_k (f, \varphi_k) \psi_k, \qquad f \in D_T,\ \varphi_k \in D_T, \tag{8}$$

where λ_k is a sequence of numbers, φ_k is an orthonormal system, and the series in (8) converges to Tf in the norm of $L_2(\Omega)$.

Theorem 3. *In order that the linear operator (8) be an integral operator of Carleman type, it is necessary and sufficient that*

for almost all $s \in \Omega$

$$\lambda^2(s) = \sum_{k=1}^{\infty} |\lambda_k|^2 |\psi_k(s)|^2 < \infty. \tag{9}$$

Necessity. By virtue of (8) and Bessel' s inequality, for almost all $s \in \Omega$

$$K^2(s) = \int_\Omega |K(s,t)|^2\, dt \geq \sum_{k=1}^{\infty} |(K(s,t)\overline{\varphi_k(t)})_t|^2 = \sum_{k=1}^{\infty} |(T\varphi_k)(s)|^2 = \lambda^2(s). \tag{10}$$

Sufficiency. Let $f \in D_T$, $\|f\| = 1$. By virtue of (8) and (9), the series

$$\sum_{k=1}^{\infty} \lambda_k (f, \varphi_k) \psi_k(s)$$

converges absolutely for almost all $s \in \Omega$ to $(Tf)(s)$. But then

$$|(Tf)(s)|^2 = \left| \sum_{k=1}^{\infty} \lambda_k (f, \varphi_k) \psi_k(s) \right|^2 \leq \sum_{k=1}^{\infty} |\lambda_k|^2 |\psi_k(s)|^2 = \lambda^2(s).$$

Consequently, $\lambda(s)$ is a majorant for T. The theorem is proved.

Like any majorant, $\lambda(s)$ satisfies inequality (6). Hence, together with (10), it follows that for almost all $s \in \Omega$

$$\lambda(s) = K(s). \tag{11}$$

Taking into account (11), Theorem 2 and Theorem 3, we arrive at the following theorem:

Theorem 4. *In order that a linear operator T of the form (8) be a Carleman integral operator, it is necessary and sufficient that:*

1) *the function $\lambda(s)$ be finite almost everywhere;*

2) $[L_2(\Omega)]_\lambda \subseteq D_{T^*}$;

3) *for any f and g from $[L_2(\Omega)]_\lambda$,*

$$(T^* f, g) = (f, T^* g).$$

Remark 1. In the case of a self-adjoint operator the second and third conditions of Theorem 4 may be omitted.

Remark 2. Theorems 1–4 are also valid for operators acting in the real Hilbert space $L_2(\Omega)$.

In conclusion let us consider an example. Let $\Omega = [0,1]$, $\chi_n^{(k)}$, $n = 0, 1, 2, \dots,$ $k = 1, 2, \dots, 2^n$, be the Haar functions, and let

$$T\chi_n^{(k)} = \alpha_n^{(k)} \chi_n^{(k)},$$

where $\alpha_n^{(k)}$ are real numbers. Let

$$a_n = \min_{1 \le k \le 2^n} |\alpha_n^{(k)}|, \qquad A_n = \max_{1 \le k \le 2^n} |\alpha_n^{(k)}|.$$

The self-adjoint operator T is a Carleman operator if the series

$$\sum_{n=0}^{\infty} 2^n A_n^2$$

converges, and is not one if the series

$$\sum_{n=0}^{\infty} 2^n a_n^2$$

diverges. Indeed,

$$\sum_{n=0}^{\infty} 2^n a_n^2 \le \lambda^2(s) \le \sum_{n=0}^{\infty} 2^n A_n^2.$$

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.