ON THE DENSITY OF GAUSSIAN DISTRIBUTIONS AND WIENER-HOPF INTEGRAL EQUATIONS

MATHEMATICS

1965

SovietRxiv

Abstract

Full Text

UDC 519.21

MATHEMATICS

Yu. A. ROZANOV

ON THE DENSITY OF GAUSSIAN DISTRIBUTIONS AND WIENER-HOPF INTEGRAL EQUATIONS

(Presented by Academician A. N. Kolmogorov, 11 IX 1965)

Let, on some measurable space $(\Omega, \mathfrak{A})$ (the space of elementary events ω), there be given a system of real measurable functions $\xi(\omega, t)$ (random variables $\xi(t)$), depending on a parameter t. Suppose that the σ-algebra $\mathfrak{A}$ is generated by the variables $\xi(t)$, i.e., coincides with the minimal σ-algebra containing all possible sets of the form $\{\xi(\omega, t) \leq x\}$. Suppose that on the σ-algebra $\mathfrak{A}$ two Gaussian distributions $P(d\omega)$ and $P_1(d\omega)$ are given, i.e., such probability measures with respect to which the joint probability distributions of the random variables $\xi(t)$ are Gaussian. Gaussian distributions are completely determined by two numerical characteristics—the mean $A(t)$ and the correlation function $B(s,t)$ of the given random variables:

$$A(t) = \int \xi(\omega, t) P(d\omega),$$

$$B(s,t) = \int [\xi(\omega, s) - A(s)][\xi(\omega, t) - A(t)] P(d\omega).$$

The question is asked under what conditions on the corresponding functions $A(t)$, $B(s,t)$ and $A_1(t)$, $B_1(s,t)$ the Gaussian distributions $P(d\omega)$ and $P_1(d\omega)$ will be mutually absolutely continuous, or, as one also says, equivalent. How can one find the density $p(\omega) = P_1(d\omega)/P(d\omega)$? These questions are of great interest in mathematical statistics, information theory, and other areas of probability theory; they are closely connected with a number of interesting problems in functional analysis and the theory of functions of a complex variable, and in recent years have attracted the attention of many mathematicians (the main results can be found, for example, in the survey article (^{1})).

Suppose the Gaussian measures $P(d\omega)$ and $P_1(d\omega)$ define probability distributions of a stationary process $\xi(t)$; then the parameter t runs over the interval $[0, T]$, and the correlation functions $B(s,t) = B(s-t)$ and $B_1(s,t) = B_1(s-t)$ depend only on the difference $s-t$ and are represented in the form

$$B(t) = \int e^{i\lambda t} F(d\lambda), \qquad B_1(t) = \int e^{i\lambda t} F_1(d\lambda),$$

where $F(d\lambda)$ and $F_1(d\lambda)$ are some positive and bounded measures on the line $-\infty < \lambda < \infty$. The solution of the question of the equivalence of $P(d\omega)$ and $P_1(d\omega)$ in the most general setting is easily reduced to the two cases considered, when either $B(t) \equiv B_1(t)$, while the functions $A(t)$ and $A_1(t)$ are arbitrary, or $A(t) \equiv A_1(t) \equiv 0$, while the correlation functions $B(t)$ and $B_1(t)$ are arbitrary.

In the relatively simple case when $B(t) \equiv B_1(t)$, the solution of the question of equivalence was given in many works and consists in the following···

Denote by $L^2_T(F)$ the class of all complex functions $\varphi(\lambda)$ on the line $-\infty < \lambda < \infty$ such that

$$\int |\varphi(\lambda)|^2 F(d\lambda) < \infty, \qquad \inf_{c_k,\, 0 \leqslant t_k \leqslant T} \int \left| \varphi(\lambda) - \sum_k c_k e^{i\lambda t_k} \right|^2 F(d\lambda) = 0.$$

For the equivalence of $P(d\omega)$ and $P_1(d\omega)$ it is necessary and sufficient that the difference $a(t) = A(t) - A_1(t)$ admit a representation of the form

$$a(t) = \int e^{i\lambda t} \varphi(\lambda) F(d\lambda), \qquad 0 \leqslant t \leqslant T, \tag{1}$$

where $\varphi(\lambda)$ is some function of the class $L^2_T(F)$. It is convenient to assume $A(t) \equiv 0$, which in no way restricts generality. Then, with respect to the distribution $P(d\omega)$, the process $\xi(t)$ admits a spectral expansion of the form

$$\xi(t) = \int e^{i\lambda t} \Phi(d\lambda)$$

(here $\Phi(d\lambda)$ is a measure with orthogonal values in $L^2(\Omega)$), and the density $p(\omega) = P_1(d\omega)/P(d\omega)$ is described by the formula

$$p(\omega) = D \exp\left\{ \int \varphi(\lambda) \Phi(d\lambda) \right\},$$

where D is a normalizing factor determined from the condition

$$\int p(\omega) P(d\omega) = 1,$$

and the function $\varphi(\lambda)$ is the same as in expression (1).

Let $A(t) \equiv A_1(t) \equiv 0$. Here, for some special cases, quite definitive results have been obtained, but there has as yet been no complete solution of the question. Such a solution, in our opinion, is proposed below. In doing so we shall assume that

$$0 < \underline{\lim}_{\lambda\to\infty} \frac{f_1(\lambda)}{f(\lambda)} \leqslant \overline{\lim}_{\lambda\to\infty} \frac{f_1(\lambda)}{f(\lambda)} < \infty, \tag{2}$$

where $f(\lambda)$ and $f_1(\lambda)$ denote the densities of the spectral measures $F(d\lambda)$ and $F_1(d\lambda)$ with respect to some positive measure $G(d\lambda)$ (for absolutely continuous spectral measures, the functions $f(\lambda)$ and $f_1(\lambda)$ are simply the spectral densities of the stationary process $\xi(t)$). We note that deviations from condition (2) hardly deserve serious attention. Instead of condition (2), without restricting generality, one may assume that for all λ

$$c \leqslant f_1(\lambda)/f(\lambda) \leqslant C$$

for some positive constants c and C.

Denote by $L^2_T(F, F_1)$ the class of all functions $\varphi(\lambda,\mu)$ in the plane $-\infty < \lambda, \mu < \infty$ such that

$$\iint |\varphi(\lambda,\mu)|^2 F(d\lambda)F_1(d\mu) < \infty,$$

$$\inf_{c_{kj},\, 0\leqslant t_k, t_j \leqslant T} \iint \left| \varphi(\lambda,\mu) - \sum_{k,j} c_{kj} e^{i(\lambda t_k - \mu t_j)} \right|^2 F(d\lambda)F_1(d\mu) = 0.$$

Theorem 1. *For the equivalence of $P(d\omega)$ and $P_1(d\omega)$ it is necessary and sufficient that the difference $b(s,t) = B(s-t) - B_1(s-t)$ of the corresponding correlation functions admit a representation of the form*

$$b(s,t) = \iint e^{-i(\lambda s - \mu t)} \varphi(\lambda,\mu) F(d\lambda) F_1(d\mu), \qquad 0 \leqslant s, t \leqslant T, \tag{3}$$

where $\varphi(\lambda,\mu)$ is some function of the class $L^2_T(F, F_1)$.
The density $p(\omega)$ is described by the formula:

$$p(\omega) = D \exp\left\{ -\frac{1}{2} \operatorname*{l.\,i.\,m.}_{n\to\infty} \left[\iint_{|\varphi|\leqslant n} \varphi(\lambda,\mu)\Phi(d\lambda)\Phi(d\mu) - \int_{|\varphi|\leqslant n} \varphi(\lambda,\lambda)F(d\lambda) \right] \right\}, \tag{4}$$

where D is the normalizing factor determined from the condition

$$\int p(\omega)P(d\omega) = 1,$$

and the function $\varphi(\lambda, \mu)$ is the same as in expression (3).

Formula (4) is obviously simplified if the integral exists

$$\int_{-\infty}^{\infty} \varphi(\lambda, \lambda)F(d\lambda).$$

Expression (3), as well as (1), is a Wiener-Hopf integral equation with respect to the unknown function $\varphi \in L_T^2$. To emphasize the importance of the study of such equations (with arbitrary functions $a(t)$ and $b(s,t)$), we note that many extremal problems of great interest for applications are reduced to their solution.

It is clear that, as an element of the corresponding space L_T^2, the solution φ of equations (1) or (3) is always unique. It is known that, for equation (1), the condition for the existence of a solution is the following: there exists a constant C such that

$$\left|\sum c(t)a(t)\right|^2 \leq C \int \left|\sum c(t)e^{i\lambda t}\right|^2 F(d\lambda)$$

for all $c(t)$, $0 \leq t \leq T$. A completely analogous existence condition for equation (3) is:

$$\left|\sum c(s,t)b(s,t)\right|^2 \leq C \iint \left|\sum c(s,t)e^{i(\lambda s - \mu t)}\right|^2 F(d\lambda)F_1(d\mu).$$

We formulate a result that opens one possible direction for the further study of equations (1) and (3), when the measure $F(d\lambda)$ is absolutely continuous and the density $f(\lambda) = F(d\lambda)/d\lambda$ has a prescribed asymptotic behavior as $\lambda \to \infty$.

Theorem 2. Suppose that for some integer n

$$0 < \underline{\lim}_{\lambda\to\infty} \lambda^{2n} f(\lambda) \leq \overline{\lim}_{\lambda\to\infty} \lambda^{2n} f(\lambda) < \infty.$$

For a solution of equation (1) to exist, it is necessary and sufficient that the function $a(t)$ have an $(n-1)$-st absolutely continuous derivative $a^{(n-1)}(t)$ and

$$\int_0^T |a^{(n)}(t)|^2 \, dt < \infty.$$

For a solution of equation (3) to exist, it is necessary and sufficient that the function $b(s,t)$ have absolutely continuous partial derivatives $\partial^{2n-1}b(s,t)/\partial s^{n-1}\partial t^n$, $\partial^{2n-1}b(s,t)/\partial s^n \partial t^{n-1}$, and

$$\int_0^T \int_0^T \left| \frac{\partial^{2n} b(s,t)}{\partial s^n \partial t^n} \right|^2 ds\, dt < \infty.$$

Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
10 IX 1965

REFERENCES

1. Yu. A. Rozanov, *Theory of Probability and Its Applications*, **9**, 3 (1964).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.