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Abstract

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ON THE m-FOLD COMPLETENESS OF HALF OF THE EIGENFUNCTIONS AND ASSOCIATED FUNCTIONS OF AN ORDINARY DIFFERENTIAL OPERATOR OF ORDER $2m$

(Presented by Academician M. V. Keldysh, July 11, 1964)

In the work (^{1}) M. V. Keldysh, along with other facts, proved the n-fold completeness of all eigenfunctions and associated functions of an ordinary differential equation with coefficients depending polynomially on the parameter.

In solving certain problems of mathematical physics and mechanics, one has to expand boundary functions not in all eigenfunctions and associated functions of the corresponding ordinary differential operator, but in half of the sets of these functions.

In the note (^{5}) we proved the completeness of half of the eigenfunctions for a non-self-adjoint differential operator of second order.

In the present work the m-fold completeness of half of the eigenfunctions and associated functions of the following non-self-adjoint differential operator of order $2m$ is proved:

$$y^{(2m)}(x) + a_1\lambda y^{2m-1}(x) + \dots + a_{2m-1}\lambda^{2m-1}y'(x) + a_{2m}\lambda^{2m}y(x) = 0, \quad (1)$$

$$y^{(k)}(0) = 0, \qquad y^{(k)}(1) = 0, \qquad k = 0, 1, \dots, m-1, \quad (2)$$

where the a_i are constants, and it is assumed that they satisfy the ellipticity conditions*.

Before proceeding to the proof of the m-fold completeness of half of the sets of eigenfunctions and associated functions of problem (1), (2), consider the following problem:

Let $\Pi \begin{pmatrix} 0 \le x < \infty \\ 0 \le y \le 1 \end{pmatrix}$ be a half-strip.

In the half-strip Π we seek a bounded solution of the following problem:

$$\mathcal{L}u \equiv \sum_{i=0}^{2m} a_i \frac{\partial^{2m} u}{\partial x^{2m-i} \partial y^i} = 0; \quad (3)$$

$$\left.\frac{\partial^k u}{\partial y^k}\right|_{y=0} = 0, \qquad \left.\frac{\partial^k u}{\partial y^k}\right|_{y=1} = 0; \tag{4}$$

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} = \varphi_k(y), \qquad k = 0, 1, \ldots, m-1, \tag{5}$$

where $\varphi_k(y)$ are given smooth functions and $a_0 = 1$. Obviously, by a linear change problem (3), (4), (5) can be reduced to the following problem:

$$\mathcal{L}u = f, \tag{6}$$

$$\left.\frac{\partial^k u}{\partial x^k}\right|_{x=0} = 0, \qquad \left.\frac{\partial^k u}{\partial y^k}\right|_{y=0} = \left.\frac{\partial^k u}{\partial y^k}\right|_{y=1} = 0, \qquad k = 0, 1, \ldots, m-1. \tag{7}$$

We shall show that problem (6), (7), and therefore also problem (3), (4), (5), has a unique solution under the assumption that f is a smooth function.

* This means that equation (3) (see below) is elliptic.

Let $u(x, y)$ and $\omega(x, y)$ be finite functions in the half-strip Π. Extend them by zero to the whole space and consider the scalar product $(\mathcal{L}u, v) = \{u, v\}$. Then, applying Parseval' s equality and taking the ellipticity condition into account, we obtain

$$(\mathcal{L}u, u) = \{u, u\} \geq \alpha \iint\limits_{\Pi} \left\{ \left(\frac{\partial^m u}{\partial x^m}\right)^2 + \left(\frac{\partial^m u}{\partial y^m}\right)^2 \right\} dx\, dy.$$

On the other hand,

$$\iint\limits_{\Pi} u^2\, dx\, dy \leq \iint\limits_{\Pi} \left(\frac{\partial^k u}{\partial y^k}\right)^2 dx\, dy, \qquad k = 1, 2, \ldots$$

Therefore

$$(\mathcal{L}u, u) = \{u, u\} \geq \alpha \iint\limits_{\Pi} \left\{ \left(\frac{\partial^m u}{\partial x^m}\right)^2 + \left(\frac{\partial^m u}{\partial y^m}\right)^2 \right\} dx\, dy \geq \alpha \iint\limits_{\Pi} u^2\, dx\, dy.$$

Completing the set of functions under consideration in the metric $W_2^{(m)}(\Pi)$, we shall show that in the resulting space the equation $\mathcal{L}u = f$ has a unique generalized solution such that $\{u, v\} = (f, v)$, where v is an arbitrary function from the space $\mathring{W}_2^{(m)}(\Pi)$. For this it is necessary to show that (f, v) is a bounded functional, and this follows from the following fact:

$$|(f,v)| \leq \|f\|_{L_2}\|v\|_{L_2} \leq \alpha\|f\|_{L_2}\|v\|_{\mathring{W}_2^{(m)}}.$$

Hence, taking Riesz’ s theorem into account, we obtain that the posed problem has a unique generalized solution.

Since equation (3) is elliptic, the generalized solution constructed in this way is a classical solution inside the domain Π and on the boundary, with the exception of the corner points. Moreover, if $f(x,y)$ is a sufficiently smooth function (i.e., the φ_k are sufficiently smooth functions), then the solution of problem (3), (4), (5) will also be a sufficiently smooth function, except at the corner points. At these two corner points, by studying specially the solution of problem (3), (4), (5), it is proved that it has derivatives up to order $(2m-1)$ inclusive with some weight.

Let

$$v(\lambda,y) = \int_0^\infty e^{-\lambda x} u(x,y)\,dx. \tag{8}$$

Obviously, $v(\lambda,y)$ is a holomorphic function of the parameter in the right half-plane.

Applying the transform (8) to problem (3), (4), (5), we have

$$\sum_{i=0}^{2m} a_i \lambda^i v^{(2m-i)} = F(\lambda,y), \tag{9}$$

$$v^{(k)}\big|_{y=0} = 0, \qquad v^{(k)}\big|_{y=1} = 0, \qquad k = 0,1,\dots,m-1, \tag{10}$$

where $F(\lambda,y)$ is a linear combination of $\varphi_k(y)$, their derivatives, and the higher derivatives of the solution of problem (3), (4), (5) at $x=0$ up to order $(2m-1)$, the existence of which was proved above.

Denote by $G(\lambda,y,\xi)$ the Green’s function of problem (9) and (10). For simplicity of the computations assume that all eigenvalues are simple and that there are no associated functions. Moreover, let $\Phi_n(y)$ be the eigenfunctions of problem (9) and (10), and let $\Psi_n(y)$ be the eigenfunctions of the adjoint problem. Then it is known (see $(^3)$) that the Green’ s function is representable in the form

$$G(\lambda,y,\xi) = \sum_{n=0}^{\infty} \sum_{j=N_n}^{N_{n+1}-1} G_j(\lambda,y,\xi), \quad \text{where} \quad G_j(\lambda,y,\xi) = \frac{\Phi_j(y)\Psi_j(\xi)}{\lambda-\lambda_j}.$$

The solution of problems (9) and (10) is written as follows:

$$v(\lambda, y) = \int_0^1 G(\lambda, y, \xi) F(\lambda, \xi)\, d\xi. \tag{10'}$$

We note that the function $G(\lambda, y, \xi)$ has zeros of order $m-1$ at the points $\xi = 0$ and $\xi = 1$, which ensures the convergence of the integral $(10')$, since the functions $F(\lambda, \xi)\xi^{m-1}$ and $F(\lambda, \xi)(1-\xi)^{m-1}$ belong to the space L_2. Taking into account the fact that $v(\lambda, y)$ is a holomorphic function of λ in the right half-plane, we obtain that for $\operatorname{Re}\lambda_n > 0$

$$\int_0^1 F(\lambda_n, \xi)\Psi_n(\xi)\, d\xi = 0.$$

Let

$$w(x, y) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{\lambda x} v(\lambda, y)\, d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{\lambda x} d\lambda \int_0^1 \sum_{n=0}^{\infty} \sum_{j=N_n}^{N_{n+1}-1} \frac{\Phi_j(y)\Psi_j(\xi)}{\lambda - \lambda_j} F(\lambda, \xi)\, d\xi. \tag{11}$$

It is known (3) that in the λ-plane one can construct a system of contours Γ_N possessing the following properties:

1°. None of the eigenvalues lies on Γ_N.

2°. Γ_N is entirely contained inside Γ_{N+1}.

3°. The shortest distance R_N from the origin to Γ_N increases without bound as N increases.

In addition, on the contours Γ_N the Green function has the estimate

$$|G(\lambda, y, \xi)| < \frac{\text{const}}{|\lambda|^{2m-1}}.$$

Then in expression (11), replacing the limits of integration from $\alpha - i\infty$ to $\alpha + i\infty$ by Γ_N, we obtain

$$w(x, y) = \frac{1}{2\pi i}\lim_{N\to\infty}\int_{\Gamma_N} e^{\lambda x} d\lambda \int_0^1 \sum_{n=0}^{\infty} \sum_{j=N_n}^{N_{n+1}-1} \frac{\Phi_j(y)\Psi_j(\xi)}{\lambda - \lambda_j} F(\lambda, \xi)\, d\xi =$$

$$= \sum_{n=0}^{\infty} \sum_{j=N_n}^{N_{n+1}-1} c_j e^{\lambda_j x}\Phi_j(y),$$

where

$$c_j = \int_0^1 F(\lambda_j, \xi)\Psi_j(\xi)\, d\xi, \qquad \operatorname{Re}\lambda_j \le 0.$$

Thus, we have

$$w(x,y) = \sum_{n=0}^{\infty} \sum_{j=N_n}^{N_{n+1}-1} c_j e^{\lambda_j x}\Phi_j(y), \qquad \operatorname{Re}\lambda_j \le 0.$$

We shall show that the function $w(x,y)$ is a solution of problem (3), (4), and (5). Clearly, $w(x,y)$ satisfies equation (3) and the boundary condition (4). We shall show that it also satisfies conditions (5). Obviously, for this it is necessary to prove that

$$\lim_{\operatorname{Re}\lambda\to\infty} \lambda v(\lambda, y) = \varphi_0(y), \tag{12}$$

$$\lim_{\operatorname{Re}\lambda\to\infty} (\lambda^{k-1}v - \lambda^k\varphi_0 - \cdots - \lambda\varphi_{k-1}) = \varphi_k(y), \qquad k = 1, 2, \cdots \tag{13}$$

Let us prove, for example, (12). To this end, divide both sides of equation (9) by λ^{2m} and denote $1/\lambda$ by ε. We obtain

$$\sum_{i=0}^{2m} a_i \varepsilon^{2m-i} v^{(2m-i)} = a_0\varepsilon\varphi_0 + \varepsilon^2(a_0\varphi_1 + a_1\varphi_0') + \cdots.$$

We seek the solution of the last equation in the form $v = v_0 + \varepsilon v_1 + \cdots$. Substituting this expression for v into the last equation and comparing terms with equal powers of ε, we have

$$v_0 \equiv 0, \qquad v_1 = \dot{\varphi}_0, \ldots$$

Consequently,

$$v = \frac{v_1}{\lambda} + \frac{v_2}{\lambda^2} + \cdots \quad \text{or} \quad \lambda v = v_1 + \frac{v_2}{\lambda} + \cdots.$$

Hence

$$\lim_{\operatorname{Re}\lambda\to\infty} \lambda v = v_1 = \varphi_0.$$

In an analogous way, (13) is also proved. It follows from this that as $x \to 0$

$$\frac{\partial^k w}{\partial x^k} \to \varphi_k, \qquad k = 0, 1, \dots, m-1.$$

Thus the following has been proved.

Theorem. *The system of eigenfunctions and associated functions of problem (1), (2), corresponding to the eigenvalues* $\{\lambda_k\}$ *for which* $\operatorname{Re}\lambda_k \leqslant 0$*, forms an* m*-fold complete system; moreover, to the system of functions* $\varphi_0, \varphi_1, \dots, \varphi_{m-1}$ *there correspond series of the form*

$$\sum_{n=0}^{\infty} \sum_{j=N_n}^{N_{n+1}-1} c_j \lambda_j^k \Phi_j, \quad k = 0, 1, \dots, m-1,$$

which are summed to the functions φ_k*, respectively, by Abel' s method.*

Let us note that from the preceding arguments there follows not only completeness, but also expansion in the eigenfunctions and associated functions of problem (1) and (2).

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Note: Figure translations are in progress. See original paper for figures.

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