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Abstract

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ON THE MOTION OF A SPATIAL DIFFUSING VORTEX TUBE IN AN INCOMPRESSIBLE VISCOUS FLUID

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The determination of the velocity field of the vortex flow arising behind a wing or the blades of a propeller, in the nonlinear formulation of the problem, reduces to considering the motion of free vortex tubes in the field of the inductive velocities created by these tubes themselves. The general problem of the motion of a free vortex volume in an ideal fluid was considered by Lichtenstein, who investigated the corresponding system of integro-differential equations ([1]). However, the practical solution of these equations for important cases of spatial distributions of vortices proves to be extremely difficult. In the present paper an approximate approach is considered to the problem of the motion of vortex tubes in a real fluid, making it possible to carry the solution through to practical applications.

Let us prescribe the position and intensity of one or several spatial vortex tubes of infinitely small cross section, located at the moment $t = 0$ in an unbounded incompressible viscous fluid, and consider the question of the further propagation of the vorticity and velocity fields of these tubes.

In a rigorous formulation, the solution of this problem is described by the Helmholtz equation

$$\frac{d\vec{\Omega}}{dt} = (\vec{\Omega} \cdot \nabla)\mathbf{V} + \nu \nabla^2 \vec{\Omega}, \tag{1}$$

to which one must add the expression for the velocity vector $\mathbf{V}$ through the vorticity vector $\vec{\Omega}$

$$\mathbf{V} = \operatorname{rot} \frac{1}{4\pi} \iiint_\tau \frac{\vec{\Omega}\, d\tau}{R}. \tag{2}$$

The integral is taken over the entire volume τ occupied at each moment by the vortices. Equations (1) and (2) describe the joint process of diffusion and transport of vorticity elements.

Assuming that each vortex tube has a cross section small in comparison with other characteristic dimensions, we shall describe approximately the diffusion and transport of the vortices on the basis of the following two assumptions:

1. The axis L of a vortex tube consists at all times of the same fluid particles moving in the field of the true inductive velocities.
2. The distribution of vorticity relative to the axis L is at each instant such as if it had been formed by diffusion from a vortex filament coinciding at that instant with L and having an intensity equal to that of the entire tube.

Let us consider the process of diffusion and transport of the vortices that follows from these assumptions.

The equation of diffusion of the vorticity field is written in the form

$$\partial\vec{\Omega}/\partial t = \nu \Delta \vec{\Omega}. \tag{3}$$

If at the initial instant of time the vorticity field is contained inside an infinitely thin vortex tube L with intensity Γ, then at subsequent instants the change of the vorticity vector in space is described by the solution of equation (3), which has the form

$$\vec{\Omega} = \int_L \frac{\Gamma}{2(\sqrt{\nu\pi t})^3}\, e^{-R^2/4\nu t}\, dl. \tag{4}$$

Here dl is the vector element of the line L; R is the distance from the point XYZ, where $\vec{\Omega}$ is evaluated, to the point of integration $\zeta\eta\xi$.

According to the assumptions indicated, we shall consider that formula (4) describes the distribution of vortices of a moving and diffusing tube if the line L of its axis is moved with the velocities of the fluid particles lying on L. To fulfill this condition, it is first of all necessary to determine the velocities induced by the vorticity field (4). The general formula (2) is suitable for this, but is excessively complicated, since in the present concrete case the question can be solved much more simply. The point is that the vorticity field (4) is a superposition of elementary fields possessing spherical symmetry, which makes it possible to reduce the three-dimensional integral (2) for this case to a one-dimensional one. The simplest way to do this is by the inverse method.

A direct verification shows that the velocity field given in the form

$$\mathbf{V} = \operatorname{rot} \int_L \frac{\Gamma}{4\pi} \frac{\chi(R)}{R}\, dl, \tag{5}$$

where $\Gamma = \text{const}$ and χ is an arbitrary sufficiently smooth function of R, has the vorticity field

$$\vec{\Omega} = -\int_L \frac{\Gamma}{4\pi} \frac{\chi''(R)}{R}\, dl, \tag{6}$$

where it is assumed that the line L has no ends.

Thus, for fields with a spatial distribution of vortices determined by (6), the velocities can be determined by evaluating the line integral (5), which may also be written in the form

$$\mathbf{V} = \int_L \frac{\Gamma}{4\pi} \frac{dl \times \mathbf{R}}{R^3}\, K(R). \tag{7}$$

Here it is denoted

$$K = \chi(R) - \chi'(R)R. \tag{8}$$

Expression (4) is a particular case of (6) and coincides with the latter if the function $\chi(R)$ is specified in the form

$$\chi(R) = \frac{2}{\sqrt{\pi}} \int_0^{R/\delta} e^{-x^2}\, dx, \tag{9}$$

where $\delta = 2\sqrt{\nu t}$.

Consequently, the induced velocities from the diffusing vortex tube considered above will be determined by formula (7), in which the function $K(R)$, according to (8) and (9), will be

$$K(R) = \frac{4}{\pi} \int_0^{R/\delta} x^2 e^{-x^2}\, dx. \tag{10}$$

Expression (7) is a generalization of the Biot-Savart formula to the case of the vortex tubes of finite cross section considered, and pere-

becomes the latter for $K(R) = 1$, when the vortex tube becomes infinitely thin. Let us note that in the particular case of a rectilinear segment AB of a diffusing vortex tube, expression (7) is integrated and gives, for the magnitude of the velocity, the expression

$$V = \frac{\Gamma}{4\pi h} \Big|{}_A^B [K(R)\cos\alpha - e^{-(h/\delta)^2} K(R\cos\alpha)] \Big|. \tag{11}$$

Here h is the distance from the point T, at which the velocity is computed, to the segment AB; R and α under the substitution sign are the lengths of the vectors $\overrightarrow{AT}$ and $\overrightarrow{BT}$, and the angles of their inclination to the vector $\overrightarrow{BA}$. Expression

(11) also passes into the known one for a segment of a vortex filament when $K(R) = 1$ and $\delta \to 0$.

Thus, at any point of the space XYZ, the velocities caused by the diffusing and moving vortex tube with axis L have been obtained in the form (7). In the particular case the point may be taken on the axis of the tube, and then (7) will determine the transport velocities of this axis. The velocities determined by (7) for all $t > 0$ are finite as a consequence of the vanishing of the quantity $K(R)$ as $R \to 0$, and the integral (7) is not singular. This circumstance follows from taking the diffusion of vortices into account and essentially distinguishes the present consideration from the case of an infinitely thin vortex filament, where these velocities have a singularity.

Using (7), let us write the equations of motion of the axis of the diffusing vortex tube. Denote by $\mathbf{l}(s,t)$ the radius vector on the axis of the tube L, corresponding to the point with Lagrangian coordinate s. At the initial instant the position of L is given, i.e. $\mathbf{l}(s,0)$ is a known function. The change of $\mathbf{l}(s,t)$ at subsequent instants of time will be described, according to the second assumption, by the integro-differential equation

$$d\mathbf{l}(s,t)/dt = \mathbf{U}(\mathbf{l}(s,t); \{\mathbf{l}(s',t)\}), \tag{12}$$

where it is denoted

$$\mathbf{U} = \int_L \frac{\Gamma}{4\pi R^3} \left[\frac{\partial \mathbf{l}(s',t)}{\partial s'} \times \mathbf{R} \right] K(R)\, ds', \qquad \mathbf{R} = \mathbf{l}(s,t) - \mathbf{l}(s',t). \tag{13}$$

In the case when the vortex tube is formed from vortices shed from a wing or blade, it is convenient to take as the parameter s the time t' of shedding of the corresponding element of the tube. Denoting by $\mathbf{B}(t)$ the radius vector of the points of shedding of the vortex element at time t, we may write the following condition imposed on the function $\mathbf{l}(t',t)$:

$$\mathbf{l}(t',t)\big|_{t=t'} = \mathbf{B}(t'). \tag{14}$$

The right-hand side of equation (12) is simultaneously a function of the point on L for which the equation is written, and a functional of the shape of the curve L. When the curve L is approximated by a finite number of points and (13) is computed approximately through the coordinates of these points, equation (12) reduces to a system of ordinary differential equations, for which the Cauchy problem must be solved.

The case in which the vortex tube under consideration is the part of the free vortices shed from a wing or blade differs from the case considered only in that, as the vortices are shed into the flow, the number of equations of the indicated system will grow with time, and at the instant when each new equation appears,

Fig. 1

Figure 1: Fig. 1

the initial Cauchy condition (14), determining at what point of space the given vortex element has been shed into the flow, will be prescribed for it.

The equations given above were applied to the calculation of the shape of a system of tip vortices of an air propeller located in an oblique flow, when the velocity vector is directed at an angle to the propeller axis close to

to 90°. The form of the vortex system obtained in this way by calculation for one of the regimes is illustrated in Fig. 1. The circulation of the tip vortices Γ was taken to be $0.02\,\omega R_1^2$, and the velocities of the flow incident on the rotor along the rotor axis and along the plane of rotation were respectively

Fig. 1. Form, obtained by calculation, of the system of vortex filaments shed from the blade tips of a helicopter rotor in horizontal flight at low speed

$0.02\,\omega R_1$ and $0.1\,\omega R_1$, where ωR_1 is the velocity of the blade tips. The tip vortices interact strongly with one another, forming a complex system of intertwining and twisting vortex lines. The induced velocities from the nonlinear system of vortices shown in Fig. 1 differ appreciably from the velocities obtained on the basis of the linear scheme.

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.