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Abstract

Full Text

Mathematics

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ON THE NUMBER OF LIMIT CYCLES ARISING FROM A SINGULAR POINT OF FOCUS OR CENTER TYPE

(Presented by Academician I. G. Petrovskii on 13 X 1964)

1°. In the present note we consider the question of the number of limit cycles of the system of differential equations

$$-\frac{dx}{dt} = b_{10}x + b_{01}y + \sum_{j+l=3} b_{jl}x^jy^l, \quad \frac{dy}{dt} = c_{10}x + c_{01}y + \sum_{j+l=3} c_{jl}x^jy^l, \quad (1)$$

arising from the singular point O ($x = 0$, $y = 0$) of the phase plane XOY , in those cases when

$$B \equiv (c_{01} - b_{10})^2 - 4(c_{10}b_{01} - b_{10}c_{01}) < 0. \quad (2)$$

Denote by E the space of coefficients of system (1) under condition (2), with the Euclidean metric, and by E_c (E_f) the set of points of E to which correspond systems (1) having the point O as a center (focus). Then $E = E_c \cup E_f$. By $S(O, \delta)$ and $S(\Sigma, \varepsilon)$ we shall denote, respectively, the δ -neighborhood of the point O of the plane XOY and the ε -neighborhood of the point Σ of the space E .

Following N. N. Bautin ^(1,2), we shall say that, for a point $\Sigma_0 \in E$, the origin O of the phase plane XOY has, relative to E , cyclicity of order k , if the following two conditions are fulfilled: a) there exist numbers $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that inside $S(\Sigma_0, \varepsilon_0)$ there is no point to which there corresponds a system of the form (1) having inside $S(O, \delta_0)$ more than k limit cycles; b) whatever positive numbers $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$ may be, there exists a point $\Sigma \in S(\Sigma_0, \varepsilon)$ to which there corresponds a system of the form (1) having k limit cycles inside $S(O, \delta)$.

When conditions a) and b) are satisfied, one also says that, under variation of the coefficients of system (1) corresponding to the point Σ_0 , k limit cycles arise from the origin of the plane XOY ⁽³⁾.

Using the method of paper ⁽²⁾, the following is proved below.

Theorem 1. *Whatever the number $k = 0, 1, 2, 3, 4, 5$ may be, in E there exists a point for which the origin of the phase plane XOY has, relative to E , cyclicity of order k . In E there are no points to which there would correspond an order of cyclicity greater than 5.*

The question of the birth of limit cycles from a singular point of the second group in the case of homogeneous polynomial nonlinear additions of the n -th degree was also considered by B. M. Peretyagin ^(4,5). The main conclusion of these works is that, in some neighborhood of the origin, no more than $n + 1$ limit cycles can appear. As Theorem 1 shows, this result is erroneous.

2°. When condition (2) is fulfilled, by a nonsingular linear transformation and division of all coefficients of the right-hand sides by $\sqrt{-B}$, one can arrange that the linear parts of system (1) take, respectively, the form $y - \lambda x$ and $x + \lambda y$. After this, by rotating the phase plane XOY one can pass to

to a system in which the coefficients of x^2y in the first equation and of xy^2 in the second are the same. Such a system can be written in the form

$$-\frac{dx}{dt} = y - \lambda x + (\omega + \theta - a)x^3 + (\eta - 3\mu)x^2y + (3\omega - 3\theta + 2a - \xi)xy^2 + (\mu - \nu)y^3, \quad (3)$$

$$\frac{dy}{dt} = x + \lambda y + (\mu + \nu)x^3 + (3\omega + 3\theta + 2a)x^2y + (\eta - 3\mu)xy^2 + (\omega - \theta - a)y^3.$$

It is easy to see that the point O always has, for system (3), relative to the space \tilde{E} of the coefficients $\lambda, a, \mu, \nu, \xi, \eta, \theta, \omega$, the same cyclicity as for system (1) under condition (2) relative to E .

We denote an arbitrary point of the space \tilde{E} by σ , the images of the sets E_c and E_f in \tilde{E} respectively by \tilde{E}_c and \tilde{E}_f , and the expression $4(\mu^2 + \theta^2) - a^2$ by \varkappa .

Lemma 1. *In each of the following three cases:*

$$\begin{aligned} 1) \lambda = \xi = a = 0; \quad 2) \lambda = \xi = \nu = \theta = 0; \\ 3) \lambda = \xi = \eta = \omega = \nu = 4(\mu^2 + \theta^2) - a^2 = 0, \end{aligned} \quad (4)$$

system (3) has a center at the origin.

The validity of this lemma is not difficult to verify. Indeed, when conditions 1) are fulfilled, the derivatives of the right-hand sides of the two equations of system (3) with respect to x and to y , respectively, coincide. In case 2), the straight line $y = -x$ is an axis of symmetry of the direction field determined by system (3) in the phase plane. In case 3), for $\mu\theta \neq 0$, as was shown by K. E. Malkin ⁽⁶⁾, system (3) has an A. M. Lyapunov integral of the form $F^2 f^{-3} = C$,

where F and f are polynomials of degrees 6 and 4. By direct differentiation it is not hard to verify that one may take

$$F = 1 - \frac{3a}{32\mu^3} \left(bu^2 + cv^2 - \frac{bu^3v}{2\mu} + \frac{b^3u^6}{96\mu^4} \right), \quad f = 1 - \frac{uv}{2\mu} + \frac{b^2u^4}{64\mu^4},$$

where $b \equiv a - 2\theta$, $c \equiv a + 2\theta$, $u \equiv cx - 2\mu y$, $v \equiv bx - 2\mu y$. For $\mu\theta = 0$ the existence of a center follows from the closedness of the set \widetilde{E}_c .

Eliminating t from system (3) and passing to polar coordinates ρ, φ , we can write the solution $\rho = \rho(\varphi)$ of the resulting equation, satisfying the initial condition $\rho(0) = \rho_0$, as a series in powers of ρ_0

$$\rho(\varphi, \sigma) = \rho_0 v_0(\varphi, \sigma) + \rho_0^3 v_1(\varphi, \sigma) + \rho_0^5 v_2(\varphi, \sigma) + \dots, \quad (5)$$

where, whatever $\varepsilon > 0$ and $\sigma_0 \in \widetilde{E}$ may be, there exists a positive number $r = r(\varepsilon, \sigma_0)$ such that the series (5) converges for all $\varphi \in [0, 2\pi]$ and $\sigma \in S(\sigma_0, \varepsilon)$ for values of ρ_0 satisfying the relation $|\rho_0| < r$.

From (5), for $\varphi = 2\pi$, we obtain the so-called succession function

$$\rho(2\pi, \sigma) = \rho_0 v_0(2\pi, \sigma) + \rho_0^3 v_1(2\pi, \sigma) + \rho_0^5 v_2(2\pi, \sigma) + \dots \quad (6)$$

The positive roots of the difference $\rho(2\pi, \sigma) - \rho_0$ correspond, obviously, to closed integral curves of system (3).

3°. The following two propositions hold:

Lemma 2. *The coefficients $v_j(2\pi, \sigma)$ ($j = 0, 1, 2, \dots$) of the succession function are entire functions of the coordinates of the point $\sigma \in \widetilde{E}$, which, for $\lambda = 0$, become homogeneous polynomials of degree j with respect to the remaining coordinates of the point σ , and $v_0(2\pi, \sigma) = \exp(2\pi\lambda)$.*

Lemma 3. *The coefficients $v_j(2\pi, \sigma)$ for $j > 0$ have the form*

$$v_j(2\pi, \sigma) = a^2\theta\kappa\Theta_j^{(6)} + a^2\theta\eta\Theta_j^{(5)} + a\theta\omega\Theta_j^{(4)} + a\nu\Theta_j^{(3)} + \xi\Theta_j^{(2)} + \lambda\Theta_j^{(1)}, \quad (7)$$

where $\Theta_j^{(l)}$ are entire functions of the coordinates of the point $\sigma \in \widetilde{E}$, with $\Theta_j^{(j+l)} = 0$ for $l > 1$, and $\Theta_j^{(j+1)}$ are nonzero real numbers.

Lemma 2 is established directly from the recurrence relations obtained for determining the functions $v_j(\varphi, \sigma)$.

To prove Lemma 3, note that, on the basis of Lemma 1, when each of the three series of conditions (4) is satisfied, all coefficients $v_j(2\pi, \sigma)$ ($j > 0$) must vanish. Hence it is easy to establish that they must have the form

$$v_j(2\pi, \sigma) = a\theta\tilde{\Theta}_j^{(6)} + a\theta\eta\tilde{\Theta}_j^{(5)} + a\theta\omega\Theta_j^{(4)} + a\nu\Theta_j^{(3)} + \xi\Theta_j^{(2)} + \lambda\Theta_j^{(1)},$$

where $\Theta_j^{(2)}$ do not contain λ ; $\Theta_j^{(3)}$ do not contain λ and ξ ; $\Theta_j^{(4)}$ do not contain $\lambda, \xi,$ and ν ; $\tilde{\Theta}_j^{(5)}$ do not contain $\lambda, \xi, \nu,$ and ω , while $\tilde{\Theta}_j^{(6)}$ do not contain $\lambda, \xi, \nu, \omega,$ and η .

To obtain formula (7), it now suffices to show that $\tilde{\Theta}_j^{(6)}$ and $\tilde{\Theta}_j^{(5)}$ contain the factor a . For this purpose it is enough to restrict ourselves to the case $\lambda = \xi = \nu = \omega = 0$ and to show that the expansion of $\rho(2\pi, \sigma) - \rho_0$ in powers of a for the system

$$\begin{aligned} dx/dt &= -y - \theta x^3 + (3\mu - \eta)x^2y + 3\theta xy^2 - \mu y^3 + a(x^3 - 2xy^2), \\ dy/dt &= x + \mu x^3 + 3\theta x^2y + (\eta - 3\mu)xy^2 - \theta y^3 + a(2x^2y - y^3) \end{aligned} \quad (8)$$

does not contain terms below the second degree.

Denoting

$$H(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1}{4}\mu x^4 + \theta x^3y + \frac{1}{2}(\eta - 3\mu)x^2y^2 - \theta xy^3 + \frac{1}{4}\mu y^4,$$

$$p(x, y) = x^3 - 2xy^2, \quad q(x, y) = 2x^2y - y^3,$$

system (8) can be rewritten in the form

$$dx/dt = -H'_y(x, y) + ap(x, y); \quad dy/dt = H'_x(x, y) + aq(x, y). \quad (9)$$

For sufficiently small h , consider the family of closed curves C_h with equation $H(x, y) = h$. Let C_{h_0} be one of these curves. In its neighborhood introduce new coordinates by the relations $H(x, y) = h_0 + \xi$, $M(x, y; s) = 0$, the latter of which, for each s , determines an arc without contact, coinciding for $s = 0$ with a segment of the half-line $\varphi = 0$, with s a cyclic coordinate subject to the condition that $ds/dt = 1$ on the curve C_{h_0} .

In the new coordinates, system (9) can be written as the equation $d\xi/ds = R(\xi, s, a)$, whose solution we seek in the form of a series in powers of a and of the initial value $\xi_0 = \xi(0)$:

$$\xi = c_{10}(s)\xi_0 + c_{01}(s)a + c_{20}(s)\xi_0^2 + c_{11}(s)\xi_0a + c_{02}(s)a^2 + \dots$$

Putting here $s = \tau$, where τ is the period on the curve C_{h_0} , we obtain, on some segment of the half-line $\varphi = 0$, the function

$$\xi = \xi_0 + c_{01}(\tau)a + c_{11}(\tau)\xi_0a + c_{02}(\tau)a^2 + \dots$$

It is not difficult to show that the requirement that the expansion of $\rho(2\pi, \sigma) - \rho_0$ in powers of a for system (8) contain no terms below the second degree is equivalent to the fulfillment of the equality $c_{01}(\tau) = 0$, independently of the choice of the curve C_{h_0} .

Let us show that this condition is fulfilled. Indeed,

$$c_{01}(\tau) = \int_0^\tau (py - qx) ds = \int_{C_{h_0}} p dy - q dx = \int_{C_{h_0}} (x^3 - 2xy^2) dy + (y^3 - 2x^2y) dx.$$

In this integral, replacing x by y and y by $-x$ entails a change of sign of the integrand. At the same time, from the equation of the curve C_{h_0} ($H(x, y) = h_0$) it is clear that, independently of the choice of h_0 , if the point with coordinates (x, y) belongs to this curve, then the point $(y, -x)$ also belongs to it.

lies on C_{h_0} . Hence it follows immediately that $C_{01}(\tau) = 0$ independently of the choice of the curve C_{h_0} from the family $H(x, y) = h$. Thus it has been proved that $v_j(2\pi, \sigma)$ for all $j > 0$ have the form (7).

The fact that $\Theta_j^{(j+1)} = 0$ for all $l > 1$, while $\Theta_j^{(j+1)}$ are constants, follows from Lemma 2. Computations show that

$$\Theta_1^{(2)} = \pi/4, \quad \Theta_2^{(3)} = -5\pi/4, \quad \Theta_3^{(4)} = 25\pi/8, \quad \Theta_4^{(5)} = 5\pi/32,$$

$$\Theta_5^{(6)} = -5\pi/96.$$

Thus Lemma 3 is proved.

4°. On the basis of Lemma 3, the function of the sequence (6) may be written as follows:

$$\begin{aligned} \rho(2\pi, \sigma) - \rho_0 &= 2\pi\lambda(1 + \lambda\Phi + \rho_0\Psi_0)\rho_0 + \Theta_1^{(2)}\xi(1 + \rho_0\Psi_1)\rho_0^3 \\ &\quad + \Theta_2^{(3)}a\nu(1 + \rho_0\Psi_2)\rho_0^5 + \Theta_3^{(4)}a\theta\omega(1 + \rho_0\Psi_3)\rho_0^7 \\ &\quad + \Theta_4^{(5)}a^2\theta\eta(1 + \rho_0\Psi_4)\rho_0^9 + \Theta_5^{(6)}a^2\theta\chi(1 + \rho_0\Psi_5)\rho_0^{11}, \end{aligned}$$

where Φ, Ψ_j are series in powers of ρ_0 with coefficients in the form of entire functions of the coordinates of the point σ .

Following the arguments of N. N. Bautin ⁽²⁾, it is now easy to establish that, whatever the point $\sigma \in \widetilde{E}$ with coordinates $\lambda_0, a_0, \mu_0, \nu_0, \xi_0, \eta_0, \theta_0, \omega_0$, one can always specify such $\varepsilon_0 > 0$ and $\delta_0 > 0$ that for all $\sigma \in S(\sigma_0, \varepsilon_0) \cap E_f$ the difference $\rho(2\pi, \sigma) - \rho_0$ has no more than 5 roots inside $(0, \delta_0)$. If, moreover, $\lambda_0 = \xi_0 = \nu_0 = \omega_0 = \eta_0 = 0$, $a_0\theta_0(4\mu_0^2 + 4\theta_0^2 - a_0^2) \neq 0$, then, whatever positive numbers $\varepsilon < \varepsilon_0$ and $\delta < \delta_0$, in $S(\sigma_0, \varepsilon)$ there exists a point σ for which system (3) has 5 limit cycles in $S(O, \delta)$. The cyclicity of the point σ_0 is equal to 4 if $\lambda_0 = \xi_0 = \nu_0 = \omega_0 = 0$, $a_0\theta_0\eta_0 \neq 0$; is equal to 3 if $\lambda_0 = \xi_0 = \nu_0 = 0$, $a_0\theta_0\omega_0 \neq 0$; is equal to 2 if $\lambda_0 = \xi_0 = 0$, $a_0\nu_0 \neq 0$; is equal to one if $\lambda_0 = 0$, $\xi_0 \neq 0$, and is equal to zero if $\lambda_0 \neq 0$. This completes the proof of Theorem 1.

Let us also note that it follows from Lemma 3 that the fulfillment of at least one of the three series of conditions (4) is necessary for the existence at the origin of a center of system (3). Using Lemma 1 and the rotation invariants of the phase plane (7), it is then easy to arrive at the conclusion that the following holds.

Theorem 2. *For the existence at the origin of a center of the equation*

$$\frac{dy}{dx} = \frac{x + (\mu + \nu)x^3 + (3\psi + 3\theta + \alpha)x^2y + (\chi + \beta)xy^2 + (\psi - \theta)y^3}{y + (\psi + \theta)x^3 + (\chi - \beta)x^2y + (3\psi - 3\theta - \gamma)xy^2 + (\mu - \nu)y^3} \quad (10)$$

it is necessary and sufficient that at least one of the following two series of conditions be satisfied:

- 1) $\alpha + \gamma = \alpha\nu + \alpha\beta + 4\beta\psi = \theta\alpha^2 + (\chi - \mu)\alpha\beta - 4\theta\beta^2 = 0$;
- 2) $\alpha + \gamma = \chi + 3\mu = \alpha + 5\psi = \beta + 5\nu = \psi^2 + 4(\nu^2 - \mu^2 - \theta^2) = 0$.

Thus we arrive at center conditions equivalent to the conditions of K. E. Malkin ⁽⁶⁾, who first established that not all cases of the existence of a center for equation (10) had been found by N. A. Sakharnikov ⁽⁸⁾, as well as by M. I. Almukhamedov ⁽⁹⁾.

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REFERENCES

1. N. N. Bautin, DAN, 24, No. 7 (1939).
2. N. N. Bautin, Mat. sborn., 30, 1 (1952).
3. N. F. Otrokov, Mat. sborn., 34, 1 (1954).

4. B. M. Perepyagin, DAN, 114, No. 1 (1957).
5. B. M. Perepyagin, Uchen. zap. Smolensk. ped. inst., 10 (1962).
6. K. E. Malkin, Volzhsk. matem. sborn., 2 (1964).
7. K. S. Sibirskii, DAN, 151, No. 3 (1963); 156, No. 2 (1964).
8. N. A. Sakharnikov, Prikl. matem. i mekh., 14, 6 (1950).
9. M. I. Almukhamedov, Izv. Kazansk. fiz.-matem. obshch., 9, 3 (1937).

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