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Abstract

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PHYSICS

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KINETIC EQUATION OF A NONIDEAL DEGENERATE BOSE GAS

(Presented by Academician N. N. Bogolyubov, 27 III 1965)

The purpose of the present work is to derive a kinetic equation for a degenerate Bose gas by means of the method of N. N. Bogolyubov and K. P. Gurov ([1]).

Consider a dynamical system of N identical spinless Bose particles, contained in a volume V, interacting pairwise with one another by a potential $\Phi(r-r')$, and described by a Hamiltonian of the form

$$H = -\int \psi^+(t,r)\left(\frac{\hbar^2}{2m}\Delta + \lambda\right)\psi(t,r)\,dr + $$
$$+\frac{1}{2}\int \Phi(r-r')\psi^+(t,r)\psi^+(t,r')\psi(t,r')\psi(t,r)\,dr'\,dr, \quad (1)$$

where $\psi(t,r)$, $\psi^+(t,r)$ are Bose operators; λ is the chemical potential.

In order to separate out the condensate part of the Bose operators, we represent them in the form

$$\psi(t,r) = \psi_1(t,r) + \varphi(t,r); \qquad \varphi(t,r) = \langle\psi(t,r)\rangle; \qquad \langle\psi_1(t,r)\rangle = 0. \quad (2)$$

Here the brackets $\langle ... \rangle$ denote a quasiaverage ([2]) with respect to the nonequilibrium density matrix. Starting from (1), (2), it is easy to write the equations of motion for the operator $\psi_1(t,r)$ and the quantity $\varphi(t,r)$. In the spatially homogeneous case considered by us, the following equalities hold:

$$\varphi(t,r) = \sqrt{(N_0/V)} = n_0^{1/2} = \text{const}; \quad (3)$$

$$\psi_1(t,r) = \frac{1}{\sqrt{V}}\sum_{f_1}{}' b_{f_1} e^{if_1 r/\hbar}, \quad (4)$$

where N_0 is the number of particles in the condensate; the sum $\sum'_{f_1}$ denotes summation over all momenta f_1 not equal to zero. This corresponds to the assumption that the creation and annihilation operators of particles with zero momentum are replaced by c-numbers. For convenience, we perform the canonical Bogolyubov $u{-}v$ transformation $b_{f_1} = u_{f_1}\xi_{f_1} + v_{f_1}\xi^+_{-f_1}$, and at the same time introduce a dimensionless small parameter ε ([3]): $v(f_1) \to \varepsilon v(f_1)$; $N_0 \to \varepsilon^{-1} N_0$.

Then the equation of motion will have the form

$$
\begin{aligned}
i\hbar\frac{\partial \xi^+_{f_1}}{\partial t} = &-(E_1^{(0)} + \Delta_1^{(1)})\xi^+_{f_1} - \varepsilon S_1(f_1)\xi_{-f_1} - \Big(n_0\frac{\varepsilon}{V}\Big)^{1/2} {\sum_{f_2}}' \{Q(f_2, f_1 - f_2; f_1)\xi^+_{f_2}\xi^+_{f_1-f_2} \\
&+ P(f_1, f_2, -(f_1+f_2))\xi_{f_2}\xi_{-(f_1+f_2)} + 2Q(f_1, f_2 - f_1; f_2)\xi^+_{f_2}\xi_{f_2-f_1}\} \\
&- \Big(\frac{\varepsilon}{V}\Big) {\sum_{f_1' f_2' f_2}}' \{W(f_1, f_2; f_2', f_1')\xi^+_{f_1'}\xi^+_{f_2'}\xi_{f_2}\Delta(f_1' + f_2' - f_2 - f_1) + \\
&+ R(f_1', f_2'; f_2, f_1)\xi_{f_1'}\xi_{f_2'}\xi_{f_2}\Delta(f_1' + f_2' + f_2 + f_1) + \\
&+ S(f_1', f_2'; f_2; f_1)\xi^+_{f_1'}\xi^+_{f_2'}\xi^+_{f_2}\Delta(f_1' + f_2' + f_2 - f_1) + \\
&+ 3S(f_1', f_2'; f_1; f_2)\xi^+_{f_2'}\xi^+_{f_2'}\xi_{f_1}\Delta(f_1' + f_2' + f_1 - f_2)\}
\end{aligned}
\tag{5}
$$

and contain two small parameters $(n_0\varepsilon/V)^{1/2}$, (ε/V), where Q, P are the matrix elements of three-particle processes; W, R, S are the matrix elements of four-particle processes;

$$
\begin{aligned}
Q(f_1', f_2'; f_1) &= \frac{1}{2}\{\gamma_{f_1';f_2'}\nu(f_1')(u_{f_1'} + v_{f_1'})(u_{f_2'}u_{f_1} + v_{f_2'}v_{f_1}) + \\
&\qquad + \nu(f_1)(u_{f_1} + v_{f_1})(u_{f_1'}v_{f_2'} + v_{f_1'}u_{f_2'})\}; \\
P(f_1', f_2', f_3') &= \frac{1}{4}\gamma_{f_1';f_2';f_3'}\nu(f_1')(u_{f_1'} + v_{f_1'})(u_{f_2'}v_{f_3'} + v_{f_2'}u_{f_3'}); \\
W(f_1', f_2'; f_1, f_2) &= \frac{1}{2}\{\gamma_{f_1';f_2'}\nu(f_2 - f_1')(u_{f_1}u_{f_2'} + v_{f_1}v_{f_2'})(u_{f_2}u_{f_1'} + v_{f_2}v_{f_1'}) + \\
&\qquad + \nu(f_1' + f_2')(u_{f_1}v_{f_2} - u_{f_2}v_{f_1})(u_{f_1'}v_{f_2'} + u_{f_2'}v_{f_1'})\}; \\
R(f_1', f_2', f_2, f_1) &= \frac{1}{6}\gamma_{f_1';f_2';f_2}\nu(f_1 + f_1')(u_{f_1}u_{f_2}v_{f_1'}v_{f_2'} + v_{f_1}v_{f_2}u_{f_1'}u_{f_2'}); \\
S(f_1', f_2', f_2; f_1) &= \frac{1}{6}\gamma_{f_1';f_2';f_2}\nu(f_1' + f_2')(u_{f_1}u_{f_2}u_{f_1'}v_{f_2'} + v_{f_1}v_{f_2}v_{f_1'}u_{f_2'}); \\
S_1(f_1) &= \Big(\frac{1}{V}\Big)\sum_{f_2}\{(\nu(0) + \nu(f_1 + f_2))2u_{f_1}v_{f_1}v^2_{f_2} + \nu(f_1 + f_2)(u^2_{f_1} + v^2_{f_1})u_{f_2}v_{f_2}\}.
\end{aligned}
\tag{6}
$$

Here $\nu(f_1)$ is a Fourier component of the potential Φ, $\gamma_{f_1;\dots;f_s} = \sum_{(p)}(+1)^P$ denotes the symmetrized sum over all permutations of s particles with momenta $f_1, \dots, f_s$. The quantity $\Delta_1^{(1)}$,

$$\Delta_1^{(1)} = -\frac{\varepsilon}{V}\sum_{f_2} W(f_1, f_2; f_2, f_1) +$$

$$+\left\{-\lambda + n_0\nu(0) + \left(\frac{\varepsilon}{V}\right)\sum_{f_2}{}' \frac{1}{2}(\nu(0) + \nu(f_2))\right\}(u_{f_1}^2 + v_{f_1}^2) \tag{7}$$

is a part of the correction to the energy of the Bogoliubov elementary excitation $E_1^{(0)}$. It is easy to find the chemical potential by means of the equation of motion for the quantity $\varphi(t, r)$, taking into account assumption (3) (2). Then, in the first approximation in ε, λ can be represented in the form

$$\lambda^{(1)} = n_0\nu(0) + \left(\frac{\varepsilon}{V}\right)\sum_{f_2}{}'(\nu(0) + \nu(f_2)) +$$

$$+\left(\frac{\varepsilon}{V}\right)\sum_{f_2}{}'\{(\nu(0) + \nu(f_2))v_{f_2}^2 + \nu(f_2)u_{f_2}v_{f_2}\}(1 + 2n_2); \qquad n_2 = \langle\xi_{f_2}^+\xi_{f_2}\rangle. \tag{8}$$

Starting from (5), we obtain the equation

$$\begin{aligned}
i\hbar\frac{\partial n_1}{\partial t} = &-\left(n_0\frac{\varepsilon}{V}\right)^{1/2}\sum_{f_2}\{Q(f_2, f_1 - f_2; f_1)(\langle\xi_{f_2}^+\xi_{f_1-f_2}^+\xi_{f_1}\rangle - \text{c. c.}) +\\
&+ P(f_1, f_2, -(f_1 + f_2))(\langle\xi_{f_1}\xi_{f_2}\xi_{-(f_1+f_2)}\rangle - \text{c. c.}) +\\
&+ 2Q(f_1, f_2 - f_1; f_2)(\langle\xi_{f_1}^+\xi_{f_2-f_1}\xi_{f_2}\rangle - \text{c. c.})\} -\\
&-\left(\frac{\varepsilon}{V}\right)\sum_{f_1'f_2'f_2}\{W(f_1, f_2; f_2', f_1')(\langle\xi_{f_1'}^+\xi_{f_2'}^+\xi_{f_2}\xi_{f_1}\rangle - \text{c. c.})\Delta(f_1 + f_2 - f_2' - f_1') +\\
&+ R(f_1', f_2', f_2, f_1)(\langle\xi_{f_1'}\xi_{f_2'}\xi_{f_2}\xi_{f_1}\rangle - \text{c. c.})\Delta(f_1' + f_2' + f_2 + f_1) +\\
&+ S(f_1', f_2', f_2; f_1)(\langle\xi_{f_1'}^+\xi_{f_2'}^+\xi_{f_2}^+\xi_{f_1}\rangle - \text{c. c.})\Delta(f_1' + f_2' + f_2 - f_1) +\\
&+ 3S(f_1', f_2', f_1; f_2)(\langle\xi_{f_2}^+\xi_{f_2'}\xi_{f_1'}\xi_{f_1}\rangle - \text{c. c.})\Delta(f_1' + f_2' + f_1 - f_2)\} -\\
&- \varepsilon S_1(f_1)(\langle\xi_{-f_1}\xi_{f_1}\rangle - \langle\xi_{f_1}^+\xi_{-f_1}^+\rangle).
\end{aligned} \tag{9}$$

According to N. N. Bogoliubov and K. P. Gurov, taking (5) into account we make the following assumption about the order of the mean quantities and about the possibility of their splitting:

$$\langle\xi^+_{f_2}\xi^+_{f_2}\xi_{f_1+f_2}\rangle = \left(n_0\frac{\varepsilon}{V}\right)^{1/2} g_1(f_1, f_2; f_1+f_2);$$

$$\langle\xi^+_{f_1}\xi^+_{f_2}\xi^+_{-(f_1+f_2)}\rangle = \left(n_0\frac{\varepsilon}{V}\right)^{1/2} g_1'(f_1, f_2, -(f_1+f_2));$$

$$\langle\xi^+_{f_1'}\xi^+_{f_2'}\xi_{f_2}\xi_{f_1}\rangle = \gamma_{f_1';f_2'} n_1 n_2 \Delta(f_1-f_2')\Delta(f_2-f_1') + \left(\frac{\varepsilon}{V}\right) g_2(f_1', f_2'; f_2, f_1);$$

$$\langle\xi^+_{f_1'}\xi^+_{f_2'}\xi_{f_2}\xi_{f_1}\rangle = \left(\frac{\varepsilon}{V}\right) g_2'(f_1', f_2, f_2, f_1); \qquad \langle\xi_{f_1}\xi^+_{f_2'}\xi^+_{f_2}\xi_{f_1}\rangle = \left(\frac{\varepsilon}{V}\right) g_2''(f_1', f_2, f_2; f_1);$$

$$\langle\xi^+_{f_1}\xi^+_{-f_1}\rangle = \varepsilon g_2'''(f_1, -f_1); \tag{10}$$

$$\left\langle\xi^+_{f_1}\xi^+_{f_2}\xi^+_{f_2'}\xi^+_{f_1'+f_3}\xi_{f_3}\right\rangle = \frac{1}{2}\left(n_0\frac{\varepsilon}{V}\right)^{1/2}\gamma_{f_1;f_2;f_2'} n_1 \Delta(f_1-f_3)\, g_1'(f_2, f_2', f_1'+f_3) + \cdots;$$

$$\left\langle\xi^+_{f_1}\xi^+_{f_2}\xi^+_{f_2'}\xi_{f_3}\xi_{-(f_1'+f_3)}\right\rangle =$$

$$= \left(n_0\frac{\varepsilon}{V}\right)^{1/2}\frac{1}{2}\gamma_{f_1;f_2;f_2'}\{n_1\Delta(f_1-f_3) g_1(f_2, f_2'; -(f_1'+f_3)) +$$

$$+ n_1\Delta(f_1+f_1'+f_3) g_1(f_2, f_2'; f_3)\} + \cdots;$$

$$\left\langle\xi^+_{f_1}\xi^+_{f_2}\xi^+_{f_2'}\xi_{f_4}\xi_{f_3'}\xi_{f_3}\right\rangle = \gamma_{f_1;f_2;f_2'} n_1 n_2 n_{2'} \Delta(f_1-f_4)\Delta(f_2-f_3')\Delta(f_2'-f_3) + \cdots,$$

where $g_1(f_1, f_2; f_1+f_2), \ldots, g_2(f_1', f_2'; f_2, f_1)$ are correlation deviations for the averages $\langle\xi^+_{f_1}\xi^+_{f_2}\xi_{f_1+f_2}\rangle, \ldots, \langle\xi^+_{f_1'}\xi^+_{f_2'}\xi_{f_2}\xi_{f_1}\rangle$ from multiplicative products of the one-quasiparticle distribution. Then equation (9) is written in the following form:

$$
\begin{aligned}
i\hbar\frac{\partial n_1}{\partial t} = & -\left(n_0\frac{\varepsilon}{V}\right)\sum_{f_2}\{Q(f_2, f_1 - f_2; f_1)(g_1(f_2, f_1 - f_2; f_1) - \text{c. c.}) + \\
& + 2Q(f_1, f_2 - f_1; f_2)(g_1(f_1, f_2 - f_1; f_2) - \text{c. c.}) + \\
& + P(f_1, f_2, -(f_1 + f_2))(g_1'(f_1, f_2, -(f_1 + f_2)) - \text{c. c.})\} \\
& - \left(\frac{\varepsilon}{V}\right)^2 \sum_{f_1' f_2' f_2}\{W(f_1, f_2; f_2', f_1')(g_2(f_1', f_2'; f_2, f_1) - \text{c. c.})\Delta(f_1' + f_2' - f_2 - f_1) + \\
& + R(f_1', f_2', f_2, f_1)(g_2^+(f_1', f_2', f_2, f_1) - \text{c. c.})\Delta(f_1' + f_2' + f_2 + f_1) + \\
& + S(f_1', f_2', f_2; f_1)(g_2''(f_1', f_2', f_2; f_1) - \text{c. c.})\Delta(f_1' + f_2' + f_2 - f_1) + \\
& + 3S(f_1', f_2', f_1; f_2)(g_2''^+(f_1', f_2', f_1; f_2) - \text{c. c.})\Delta(f_1' + f_2' + f_1 - f_2)\} \\
& - \varepsilon^2 S_1(f_1)(g_2'''^+(f_1, -f_1) - \text{c. c.}).
\end{aligned}
\tag{11}
$$

Obviously, in order to obtain the kinetic equation with accuracy up to and including terms of order ε^2, it is necessary to find all the correlation deviations g_2 in the zeroth approximation and all the quantities g_1 in the first approximation. Since the chain of coupled equations for g_1 and g_2 is too complicated, we shall not present it.

For simplicity, in what follows we shall consider only the lowest order of the three- and four-particle processes and shall neglect all higher-order processes, except for the proper energy part. On this basis, using the decoupling method (10), the initial condition of weakening of correlations $t \to -\infty$: $g_1, g_1' \to 0$, $g_2, g_2', g_2'', g_2''' \to 0$, and the assumption that n_1 varies slowly in time in comparison with the variation of the correlation deviations g_1, g_1' and $g_2, g_2', g_2'', \ldots$ ([4]), it is easy to find the solution g_1, g_1' and $g_2, g_2', \ldots$ in the adopted approximation and to obtain the final form of the kinetic equation

$$i\hbar\frac{\partial n_1}{\partial t} = 2\left(n_0\frac{\varepsilon}{V}\right)\sum_{f_2}\Bigg\{|Q(f_2, f_1 - f_2; f_1)|^2\times$$
$$\times\left[\frac{(1+n_2)(1+n_{1-2})n_1 - n_2 n_{1-2}(1+n_1)}{(E_2^{(1)} + E_{1-2}^{(1)} - E_1^{(1)}) + i(s + \gamma_2 + \gamma_{1-2} + \gamma_1)} - \text{c. c.}\right] +$$
$$+\left|P(f_1, f_2, -(f_1+f_2))\right|^2\times$$
$$\times\left[\frac{n_1 n_2 n_{-(1+2)} - (1+n_1)(1+n_2)(1+n_{-(1+2)})}{(E_1^{(1)} + E_2^{(1)} + E_{-(1+2)}^{(1)}) + i(s + \gamma_1 + \gamma_2 + \gamma_{-(1+2)})} - \text{c. c.}\right] +$$
$$+2|Q(f_1, f_2 - f_1; f_2)|^2\times$$
$$\times\left[\frac{n_1 n_{2-1}(1+n_2) - (1+n_1)(1+n_{2-1})n_2}{(E_1^{(1)} + E_{2-1}^{(1)} - E_2^{(1)}) + i(s + \gamma_1 + \gamma_{2-1} + \gamma_2)} - \text{c. c.}\right]\Bigg\} +$$
$$+\left(\frac{\varepsilon}{V}\right)^2\sum_{f_1' f_2' f_2}\Bigg\{2|W(f_1', f_2'; f_2, f_1)|^2\left(\frac{2\pi}{i}\right)\delta(E_1^{(0)} + E_2^{(0)} - E_2^{(0)} - E_1^{(0)})\times$$
$$\times[(1+n_{1'})(1+n_{2'})n_2 n_1 - n_{1'}n_{2'}(1+n_2)(1+n_1)]\Delta(f_1' + f_2' - f_2 - f_1) +$$

$$+6|R(f_1', f_2', f_2, f_1)|^2\left(\frac{2\pi}{i}\right)\delta\left(E_{1'}^{(0)} + E_{2'}^{(0)} + E_2^{(0)} + E_1^{(0)}\right)\times$$
$$\times\left[n_{1'}n_{2'}n_2 n_1 - (1+n_{1'})(1+n_{2'})(1+n_2)(1+n_1)\right]\Delta(f_1' + f_2' + f_2 + f_1) +$$
$$+6|S(f_1', f_2', f_2; f_1)|^2\left(\frac{2\pi}{i}\right)\delta\left(E_{1'}^{(0)} + E_{2'}^{(0)} + E_2^{(0)} - E_1^{(0)}\right)\times$$
$$\times\left[(1+n_{1'})(1+n_{2'})(1+n_2)n_1 - n_{1'}n_{2'}n_2(1+n_1)\right]\Delta(f_1' + f_2' + f_2 - f_1) +$$
$$+18|S(f_1', f_2', f_1; f_2)|^2\left(\frac{2\pi}{i}\right)\delta\left(E_{1'}^{(0)} + E_{2'}^{(0)} + E_1^{(0)} - E_2^{(0)}\right)\times$$
$$\times\left[n_{1'}n_{2'}n_1(1+n_2) - (1+n_{1'})(1+n_{2'})(1+n_1)n_2\right]\Delta(f_1' + f_2' + f_1 - f_2)\Bigg\} +$$
$$+\varepsilon^2|S_1(f_1)|^2\left(\frac{2\pi}{i}\right)\delta\left(E_1^{(0)} + E_{-1}^{(0)}\right)\left[n_1 n_{-1} - (1+n_1)(1+n_{-1})\right], \tag{12}$$

where

$$s \to 0_+; \qquad E_1^{(1)} = E_1^{(0)} + \Delta_1^{(1)} + \Delta_2'^{(1)};$$

$$\Delta'^{(1)} = -\left(\frac{\varepsilon}{V}\right)\sum_{f_1'} 2n_1' W(f_1, f_1'; f_1', f_1) - 2\left(n_0\frac{\varepsilon}{V}\right)\sum_{f_1', f_2'}\Bigg\{|Q(f_1', f_2'; f_1)|^2\times$$
$$\times\frac{(1+n_{1'}+n_{2'})}{(E_{1'}^{(0)}+E_{2'}^{(0)}+E_1^{(0)})}\Delta(f_1'+f_2'-f_1) + |P(f_1', f_2', f_1)|^2\frac{(1+n_{1'}+n_{2'})}{(E_{1'}^{(0)}+E_{2'}^{(0)}+E_1^{(0)})}\times$$
$$\times\Delta(f_1'+f_2'+f_1) + 2|Q(f_1, f_1'; f_2')|^2\frac{(n_{2'}-n_{1'})}{(E_1^{(0)}+E_{1'}^{(0)}-E_{2'}^{(0)})}\Delta(f_1+f_1'-f_2')\Bigg\}; \tag{13}$$

$$\gamma_1 = 2\pi\left(n_0\frac{\varepsilon}{V}\right)\sum_{f_1' f_2'}\Big\{|Q(f_1', f_2'; f_1)|^2(1+n_{1'}+n_{2'})\delta(E_{1'}^{(0)}+E_{2'}^{(0)}-E_1^{(0)})\times$$
$$\times\Delta(f_1'+f_2'-f_1) + |P(f_1', f_2', f_1)|^2(1+n_{1'}+n_{2'})\delta(E_{1'}^{(0)}+E_{2'}^{(0)}+E_1^{(0)})\times$$
$$\times\Delta(f_1'+f_2'+f_1) + 2|Q(f_1, f_1'; f_2')|^2(n_{1'}-n_{2'})\times$$
$$\times\delta(E_1^{(0)}+E_{1'}^{(0)}-E_{2'}^{(0)})\Delta(f_1+f_1'-f_2')\Big\}, \tag{14}$$

where γ_1 is a positive quantity. Its reciprocal is the lifetime of the state of quasiparticles with momentum f_1; it is associated with quasiparticle damping.

From this it is easy to observe that, in the first approximation in ε, expression (12) coincides with the result of N. N. Bogoliubov ([5]). In this case the kinetic equation has the standard form of the Boltzmann equation. In the general case the kinetic equation we have obtained differs from the result of work ([5]) in the following respects: first, four-quasiparticle processes appear, which make important contributions to the theory of relaxation of helium II ([6]); second, the δ-functions are broadened, which is connected with the appearance of the damping γ. They mean that in scattering, strictly speaking, energy is not conserved—this is one of the important retardation effects in kinetic equations, as was noted in work ([7]).

In conclusion we note that, using (6), (7), (8), and (13), it is easy to verify that as $f_1 \to 0$ the correction $\Delta_1^{(1)} + \Delta_1'^{(1)}$ to the spectrum of elementary excitation also tends to zero. This means that there is no gap in the elementary-excitation spectrum of the Bose system.

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Note: Figure translations are in progress. See original paper for figures.