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Abstract

Full Text

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ON THE UNIVALENCE OF A CLASS OF MEROMORPHIC FUNCTIONS

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As is known, an analytic function is meromorphic if, at finite distance, it has no singular points other than poles. Consider the class of meromorphic functions

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{(z - a_k)^m}, \tag{1}$$

where $m \geqslant 1$ is an arbitrary natural number; A_k and a_k, $k = 1, 2, \dots, n$, are two systems of constant complex numbers, none of which is equal to zero. Restrictions may be imposed on the poles a_k with respect to their position. Below we give some investigations connected with this problem. Under the condition that the argument of the ratio A_k/a_k^{m+1}, $k = 1, 2, \dots, n$, is constant, we shall prove the following theorem.

Theorem. Each function of the form (1) is univalent in the closed circular domain

$$|z| \leqslant \delta \sin \frac{\pi}{2(m+1)}, \tag{2}$$

where δ is the distance of the nearest pole a_k from the origin.

For the proof of the theorem we shall use the following lemma.

Lemma. The real part of the product

$$F(z_1, z_2, \dots, z_n) = (1 - z_1)^{m_1}(1 - z_2)^{m_2} \dots (1 - z_n)^{m_n}, \quad n \geqslant 1, \tag{3}$$

where throughout the principal values of the power functions are taken and the real exponents $|m_k| \geqslant 1$, $k = 1, 2, \dots, n$, are arbitrary, is always positive, i.e.

$$\operatorname{Re} F(z_1, z_2, \dots, z_n) > 0, \tag{4}$$

when the complex numbers $z_1, z_2, \dots, z_n$ vary arbitrarily in the disk

$$|z| < \sin\frac{\pi}{2n\max|m_k|}. \tag{5}$$

The number $\sin\dfrac{\pi}{2n\max|m_k|}$ cannot be replaced by a larger number.

Proof. We note that the absolute value $|\arg(1-z_k)|$ does not exceed the magnitude of the acute angle $\dfrac{\pi}{2n|m_k|}$ between the tangent from the point $z=1$ to the circle $|z| = \sin\dfrac{\pi}{2n|m_k|}$ and the real axis, when z_k varies inside and on this circle, i.e.

$$|\arg(1-z_k)| \leqslant \frac{\pi}{2n|m_k|},$$

with equality attained only when z_k lies at the point of tangency. From

from the equality $\arg F(z_1, z_2, \dots, z_n) = \sum_{k=1}^{n} m_k \arg(1-z_k)$ it follows that $|\arg F| \leqslant \dfrac{\pi}{2}$, and equality is obtained only for

$$z_k = \sin\frac{\pi}{2n|m_k|}\left|\sin\frac{\pi}{2n|m_k|} \pm i\cos\frac{\pi}{2n|m_k|}\right|, \quad k = 1, 2, \dots, n.$$

The minus sign corresponds to the second point of tangency. Only in this exceptional case is the product $F(z_1, z_2, \dots, z_n)$ a purely imaginary number. If one replaces the disk $|z| < \sin\dfrac{\pi}{2n|m_k|}$ by the smallest one, i.e. by the disk

$$|z| < \sin\frac{\pi}{2n\max|m_k|},$$

then, under an arbitrary variation of the numbers $z_1, z_2, \dots, z_n$ in it, one always has $\operatorname{Re} F > 0$. It is clear that this disk is the largest with this property.

For the proof of the theorem let us note that if $z_1 \neq z_2$ are two points of the closed disk

$$|z| \leqslant \delta \sin\frac{\pi}{2(m+1)},$$

then the difference

$$f(z_2) - f(z_1) = -m\sum_{k=1}^{n} A_k \int_{z_1}^{z_2} \frac{dt}{(z-a_k)^{m+1}},$$

where the integration may be carried out along the straight segment $z_1 z_2$. When the real variable t increases from 0 to 1, the variable of integration

$$z = (1-t)z_1 + tz_2$$

describes this segment and, consequently,

$$-\frac{1}{m}\frac{f(z_2)-f(z_1)}{z_2-z_1} = (-1)^{m+1}\sum_{k=1}^{n}\frac{A_k}{a_k^{m+1}}\int_0^1 \frac{dt}{\left(1-\frac{z}{a_k}\right)^{m+1}}, \quad z=(1-t)z_1+tz_2. \tag{7}$$

Under the assumptions made, the real part of the integrand is positive in the interval $0 \leqslant t \leqslant 1$, or possibly is equal to zero at its endpoints, because, according to the lemma, for $n = 1$, $m_1 = m + 1$, this is so when

$$\left|\frac{z}{a_k}\right| \leqslant \frac{|z|}{\delta} \leqslant \sin\frac{\pi}{2(m+1)}.$$

Only in the case when $|a_k| = \delta$ and at least one of the cases

$$z_{1,2} = \delta\sin\frac{\pi}{2(m+1)}\left(\sin\frac{\pi}{2(m+1)} \pm i\cos\frac{\pi}{2(m+1)}\right),$$

holds, for $t = 0$ or 1, is the real part equal to zero. Consequently, the real parts of the integrals in (7) are positive. From the condition

$$\arg\frac{A_k}{a_k^{m+1}} = \text{const}$$

we obtain that all terms on the right-hand side of (7) lie on one side of the straight line passing through the origin (the one obtained by rotating the imaginary axis through the angle

$$\arg\frac{A_k}{a_k^{m+1}}$$

).

Consequently, the right-hand side of (7) is not equal to zero.

This proves the theorem.

The results obtained also hold for a broader class of functions

$$f(z) = \sum_{k=1}^{n}\frac{A_k}{(z-a_k)^{m_k}(z-b_k)^{p_k}(z-c_k)^{q_k}\ldots}$$

under suitable conditions on the numbers

$$A_k,\ a_k,\ b_k,\ c_k, \ldots,\ m_k,\ p_k,\ q_k, \ldots$$

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Note: Figure translations are in progress. See original paper for figures.