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Abstract

Full Text

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On Attracting and Attracted Sets

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Let T be a continuous single-valued mapping of a compact space E into itself. Every point $x \in E$ generates the iterative sequence $\{T^j x\}_{j=0}^{\infty}$. If the points $x, Tx, \dots, T^{k-1}x$ are pairwise distinct and $T^{k-1}x = x$, then the points $x, Tx, \dots, T^{k-1}x$ form a cycle of order k.

A point y is called an ω-limit point of the sequence $\{T^j x\}$ if, for every neighborhood U of the point y and every $n > 0$, there exists a number $m \geq n$ such that $T^m x \in U$. We shall denote the set of ω-limit points of the sequence $\{T^j x\}$ by Ω_x.

1. The set $\Omega = \Omega_x$ is closed and $T\Omega = \Omega$. The following two theorems characterize the mapping T on the set Ω.

Theorem 1. If U is a set open in Ω, and $U \neq \Omega$, then the closure of the set TU is not contained in U.

From Theorem 1 it follows immediately that

Corollary 1. If $\Omega' \subset \Omega$ is such that $T\Omega' = \Omega'$, then Ω' cannot be both closed and open in Ω.

Corollary 2. If Ω is finite, the points of the set Ω form a cycle.

Corollary 3. If Ω is infinite, every point of a cycle belonging to Ω is a limit point for the points of the set Ω.

Theorem 2. If the set Ω is infinite and $T^j x \in \Omega$ for $j \geq j_0$, then for any set U open in Ω, the derived set of the set

$$\bigcup_{j=0}^{\infty} T^j U$$

is Ω.

If the set Ω has interior points in E, then a number j_0 such that $T^j x \in \Omega$ for $j \geq j_0$ always exists.

Theorems 1 and 2 completely characterize the mapping on the set Ω. By this is meant the following.

A. If a continuous mapping T of a compact space E into itself is given such that on the closed set Ω, $T\Omega = \Omega$, and the assertion of Theorem 2 holds, then there exists a point $x \in \Omega$ such that $\Omega_x = \Omega$.

B. If a closed set Ω has no interior points in E, contains no isolated points of E, and a continuous mapping T is given on it such that $T\Omega = \Omega$ and Theorem 1 holds, then the mapping T can be extended to a closed set E', $\Omega \subset E' \subseteq E$, so that the mapping T on E' is continuous and there exists a point $x \in E'$ for which $\Omega_x = \Omega$.

The question of the structure of the set of ω-limit points of an iterative sequence is reduced to the following: what is the structure of a closed set $\Omega \subseteq E$, if on the set Ω one can define a continuous mapping T such that $T\Omega = \Omega$ and Theorem 1 or 2 holds.

One may note the following result: if the compact space E is locally connected and the set Ω has interior points in E, then

$$\Omega = \bigcup_{j=1}^{k} \Omega^{(j)}, \quad 1 \leq k < \infty,$$

where $\Omega^{(1)}, \dots, \Omega^{(k)}$ are connected closed sets having no common points, and $T\Omega^{(j)} = \Omega^{(j+1)}$, $j = 1, 2, \dots, k-1$, $T\Omega^{(k)} = \Omega^{(1)}$.

2. We shall say that a point $x \in E$ is **attracted by** a set Ω if Ω is the set of ω-limit points of the sequence

$$\{T^j x\}_{j=0}^{\infty}.$$

The set consisting of the points of the compactum that are attracted to a given set Ω will be denoted by $P(\Omega)$. In what follows we shall agree to denote by Ω (with indices or without them) only those sets for which $P(\Omega)$ is nonempty.

Theorem 3.

$$\bigcup_{\Omega' \supseteq \Omega} P(\Omega')$$

is a set of type G_δ.

Let Σ be a system of open sets σ_i, $i = 1, 2, \dots$, such that: 1) $\sigma_i \cap \Omega \neq 0$, $i = 1, 2, \dots$; 2) for every neighborhood U of a point $x \in \Omega$ there is a $\sigma \in \Sigma$ contained in U. For each $\sigma_i \in \Sigma$ construct the open set $S(\sigma_i)$, consisting of all preimages of the set σ_i: $x \in S(\sigma_i)$ if and only if there exists $k \geq 0$ such that

$$T^k x \in \sigma_i.$$

Then

$$\bigcap_{i=1}^{\infty} S(\sigma_i) = \bigcup_{\Omega' \supseteq \Omega} P(\Omega').$$

Indeed, if $x \in P(\Omega')$, where $\Omega' \supseteq \Omega$, then $x \in S(\sigma_i)$, $i = 1, 2, \ldots.$ If $x \in P(\Omega'')$ and Ω'' does not contain Ω, i.e. there exists a point $y \in \Omega$ not belonging to Ω'', then there is a $\delta_i \ni y$ containing not a single point of the sequence $\{T^j x\}_{j=0}^{\infty}$. Consequently,

$$x \notin S(\sigma_i)$$

and

$$x \notin \bigcup_{\Omega' \supseteq \Omega} P(\Omega').$$

Theorem 4.

$$\bigcup_{\Omega' \subseteq \Omega} P(\Omega')$$

is a set of type $F_{\sigma\delta}$.

Indeed, if F is an arbitrary closed set, then the set $p(F)$, consisting of the points $x \in E$ for which $T^j x \in F$ for $j \geq j_0$ (j_0 depends on x), is a set of type F_σ. Consider a sequence of open sets

$$U_1 \supset U_2 \supset U_3 \supset \ldots$$

such that

$$\bigcap_{i=1}^{\infty} U_i = \Omega,$$

and let F_i be the closures of the sets U_i. Construct the sets $p(F_i)$, $i = 1, 2, \ldots.$

$$\bigcap_{i=1}^{\infty} p(F_i)$$

is the set

$$\bigcup_{\Omega' \subseteq \Omega} P(\Omega').$$

Indeed, if $x \in P(\Omega')$, $\Omega' \subseteq \Omega$, then $x \in p(F_i)$, $i = 1, 2, \ldots.$ If $x \in P(\Omega'')$ and there exists a point $y \in \Omega''$ not belonging to Ω, then there is a number i_0 such that $y \notin F_{i_0}$, and then $x \notin p(F_i)$, $i \geq i_0$.

Corollary. $P(\Omega)$ is a set of type $F_{\sigma\delta}$.

If the set $\Omega \cap P(\Omega)$ is nonempty, then $P(\Omega)$ on the set Ω, as follows from Theorem 3, is a set of type G_δ of the second category.

The question arises whether there exist mappings for which the sets $P(\Omega)$ are sets of type G_δ or F_σ and are not sets of a simpler type. An affirmative answer to this question is given by the theorems formulated below.

3. Consider the case when E is a segment of the real line.

Theorem 5. If Ω is infinite, then $P(\Omega)$ is a set of class ≥ 1 in the Baire–Wali-Poussin classification.

Corollary. If Ω is infinite and there is no set $\Omega' \supset \Omega$, then $P(\Omega)$ is G_δ and is not F_σ.

Theorem 5 follows from Lemma 1.

Lemma 1. If the hypotheses of Theorem 5 are fulfilled, then: 1) in every neighborhood of each point $x \in P(\Omega)$ there are points $x' < x,\ x' \notin P(\Omega)$ (if x is not the left endpoint of E) and $x'' > x,\ x'' \notin P(\Omega)$ (if x is not the right endpoint of E); 2) if an interval $(a, b) \subset P(\Omega)$, then also the points $a, b \in P(\Omega)$.

Theorem 6. If Ω contains a cycle and there exists $\Omega' \supset \Omega$, then

$$\bigcup_{\Omega' \supseteq \Omega} P(\Omega')$$

is a set of the second class.

The proof of Theorem 6 is based on Lemmas 1 and 2.

Lemma 2. *If* $(a, b) \subset \bigcup_{\Omega' \supseteq \Omega} P(\Omega')$ *and* Ω *contains a cycle, then there exists an* Ω' *such that* $(a, b) \subset P(\Omega')$.

Lemma 2 is apparently also true if Ω does not contain cycles.

Theorem 7. *If* Ω *contains a cycle and 1) there exists* $\Omega' \supset \Omega$; *2) in every neighborhood* U *of the set* Ω *there is a point* $x \in P(\Omega),\ x \notin \Omega,\ T^j x \in U$, $j = 0, 1, 2, \ldots$, *then the set* $P(\Omega)$ *is a set of the third class in the Baire–Vallée-Poussin classification.*

Thus, under the conditions of the theorem, the set $P(\Omega)$ is $F_{\sigma\delta}$ and is not $G_{\delta\sigma}$.

For example, for the mapping $Tx = x - x \sin \frac{1}{x}$ of the interval $[0, 1]$, the set of points x for which $T^j x \to 0$ as $j \to \infty$ is precisely $F_{\sigma\delta}$.

If the set Ω is infinite and contains at least one isolated point (and hence also a countable set of isolated points), then condition 2) of the theorem is satisfied.

The outline of the proof of Theorem 7 is as follows: 1) in the set

$$\bigcup_{\Omega' \supseteq \Omega} P(\Omega')$$

one chooses a subset J homeomorphic to the set of irrational numbers ([1]); 2) it is shown that the set $P(\Omega) \cap J$ can be obtained in the same way and from the same elements as a Baire set of the third class ([2]).

As a consequence of Theorems 6 and 7 we obtain Theorem 8.

Theorem 8. *If the conditions of Theorem 7 are fulfilled,*

$$\bigcup_{\Omega' \supseteq \Omega} P(\Omega')$$

is a set of the third class.

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CITED LITERATURE

[1] P. S. Aleksandrov, P. S. Uryson, *Math. Ann.*, **98** (1927). [2] N. N. Luzin, *Lectures on Analytic Sets*, Collected Works, **2**, Publishing House of the Academy of Sciences of the USSR, 1958, Ch. II.

Note: Figure translations are in progress. See original paper for figures.