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1965

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Abstract

Full Text

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ON THE GROWTH OF FUNCTIONS CONJUGATE TO ENTIRE FUNCTIONS OF FINITE DEGREE

(Presented by Academician S. N. Bernstein, 1 IX 1964)

Let W^2 denote the Wiener space of functions $f(x)$, measurable in the sense of Lebesgue, defined on the real axis and satisfying the condition

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2}\, dx < \infty. \tag{1}$$

If, for a function $f(x) \in W^2$, for some values of the numbers (in general, complex) a, b,

$$\int_{-\infty}^{\infty} \frac{|f(x)-a|^2}{|x-b|^2}\, dx < \infty,$$

then, relying on Plancherel' s theorem and using the Fourier transform $F^*(x)$ of the fraction $F_1(x) = f(x) - a/x - b$, one can define the function $\tilde{f}(x)$ conjugate to $f(x)$ by the formula (see ([2]), p. 69)

$$\tilde{f}(x) = \frac{x-b}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} F^*(t)\, \operatorname{sign} t \cdot \frac{1-e^{ixt}}{t}\, dt.$$

In particular, such a definition of the conjugate function with $a = f(0)$, $b = 0$ is applicable to all functions belonging to W^2.

In the present note we consider functions, defined in this way, that are conjugate to entire functions of exponential type from W^2.

Let, for some $\sigma > 0$, as is customary, B_σ denote the class, introduced by S. N. Bernstein ([1]), of entire functions of degree $\leq \sigma$, bounded on the real axis. For a function $f(x) \in B_\sigma$ the following representations, resulting from the Wiener-Paley theorem, are known (see ([2]), p. 73):

$$f(x) = f(0) + \frac{x}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \Psi(u) e^{iux}\, dx, \tag{2}$$

$$\tilde{f}(x) = \frac{x}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} i \operatorname{sign} u \cdot e^{iux} \Psi(u)\, du, \tag{3}$$

in which $\Psi(u)$ is an arbitrary measurable function satisfying the condition

$$\int_{-\sigma}^{\sigma} |\Psi(u)|^2 \, du < \infty.$$

These representations, which are valid for all entire functions of degree $\leq \sigma$ belonging to W^2, show that if an entire function $f(x)$ of degree $\leq \sigma$ belongs to W^2, then $\tilde{f}(x)$ is also an entire function of degree $\leq \sigma$, belonging to the space W^2.

Consequently, for any entire function $f(x)$ of exponential type satisfying condition (1), the function $\tilde{f}(x)$ conjugate to it grows on the real axis, as $|x| \to \infty$, more slowly than $|x|$, and in the general case this conclusion admits no refinement.

In connection with this, the question arises of the possibility of obtaining a more precise estimate for the modulus of the conjugate function $\tilde{f}(x)$ in the case when $f(x) \in B_\sigma$. The answer to this question is given by the following

Theorem 1. *If an entire function $f(x)$ of degree not exceeding $\sigma > 0$ satisfies everywhere on the real axis the condition $|f(x)| \leqslant 1$, then for the function $\tilde{f}(x)$ conjugate to it the inequality*

$$|\tilde{f}(x)| \leqslant \frac{4}{\pi} \ln(|x| + 1) + O(1) \tag{4}$$

holds.

Inequality (4) shows that, instead of the general estimate $\tilde{f}(x) = o(|x|)$ as $|x| \to \infty$, which cannot be improved on the class of all entire functions of degree $\leqslant \sigma$ from W^2, in the case when $f(x) \in B_\sigma$ a more precise estimate is valid, in which the majorant for $|\tilde{f}(x)|$ already has order of growth $\ln |x|$. This result on the whole class B_σ, in a certain sense, cannot be further refined.

Theorem 2. *Whatever function $\varepsilon(x) > 0$, defined for $-\infty < x < \infty$, is such that*

$$\lim_{|x| \to \infty} \varepsilon(x) = 0,$$

there exists an entire function $f(x) \in B_\sigma$ such that

$$\overline{\lim_{|x| \to \infty}} \frac{|\tilde{f}(x)|}{\varepsilon(x) \ln(|x| + 1)} = \infty. \tag{5}$$

Theorem 2 confirms the circumstance that the function $\tilde{f}(x)$ conjugate to a function $f(x) \in B_\sigma$ may turn out to be unbounded on the real axis, and also shows that the estimate of growth given above,

$$|\tilde{f}(x)| = O\{\ln(|x| + 1)\},$$

on the whole class B_σ cannot be replaced by a better, in order, estimate

$$O\{\varepsilon(x)\ln(|x|+1)\},$$

where $\varepsilon(x) > 0$ and $\lim_{|x|\to\infty}\varepsilon(x) = 0$.

Consider the class $W_\sigma^{(1)}$ of entire functions $f(x)$ of degree not exceeding σ, satisfying condition (1) and such that

$$\left|f\left\{(2k+1)\frac{\pi}{\sigma}\right\}\right| \leqslant 1 \quad (k = 0, \pm1, \pm2, \pm3, \ldots), \tag{6}$$

and also the class $W_\sigma^{(2)}$ of entire functions $f(x)$ of degree $\leqslant \sigma$, satisfying condition (1) and the condition

$$\left|f\left(\frac{2k\pi}{\sigma}\right)\right| \leqslant 1 \quad (k = 0, \pm1, \pm2, \pm3, \ldots). \tag{7}$$

Along with the theorems stated above for the class B_σ, we note the following proposition concerning conjugate functions for the classes $W_\sigma^{(1)}$ and $W_\sigma^{(2)}$.

Theorem 3. *For functions conjugate to functions of the classes $W_\sigma^{(1)}$ and $W_\sigma^{(2)}$, the equalities*

$$\sup_{f\in W_\sigma^{(1)}} \tilde{f}\left(\frac{2k\pi}{\sigma}\right) = \frac{4}{\pi}\ln(|k|+1) + O\{1\}, \tag{8}$$

$$\sup_{f\in W_\sigma^{(2)}} \tilde{f}\left(\frac{2k+1}{2\sigma}\pi\right) = \frac{4}{\pi}\ln(|k|+1) + O(1). \tag{9}$$

are valid.

For the classes $W_\sigma^{(1)}$ and $W_\sigma^{(2)}$ propositions analogous to Theorem 3 are valid.

Received
10 VII 1964

REFERENCES

1. S. N. Bernstein, **S. R.**, **176**, 1603 (1923).

2. N. I. Akhiezer, *Lectures on the Theory of Approximation*, 1947.

Note: Figure translations are in progress. See original paper for figures.