

On the Best Approximation of the Function $\left((x-c)^{r-1} |x-c|^{1+\alpha} \right)$ by Polynomials in the Metric of the Space $(L_q(-1,1))$ $(q \geq 1)$

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Abstract

Full Text

Mathematics

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On the Best Approximation of the Function $(x - c)^{r-1}|x - c|^{1+\alpha}$ by Polynomials in the Metric of the Space $L_q(-1, 1)$ ($q \geq 1$)

(Presented by Academician S. N. Bernstein on 16 II 1965)

Owing to the investigations of S. N. Bernstein ⁽¹⁻³⁾, the asymptotic properties of the best uniform approximation of the function $|x - c|^p$, which is the simplest function having an algebraic singularity, are now well known. These works play a fundamental role in the study of the asymptotic properties of the best approximation by polynomials of other functions with singularities of the same type.

Subsequently, in the works of S. M. Nikol'skii ^(4,5), the asymptotic properties of the best approximation of the function $|a - x|^s$ by polynomials in the metric of the space L on the interval $[-1, 1]$ were considered. In investigations concerning the case of the metric L , considerations connected with A. A. Markov's criterion played an important role.

The present note is devoted to the best approximation by algebraic polynomials in the metric L_q on $[-1, 1]$ for arbitrary $q \geq 1$ (except for the cases $q = \infty$ and $q = 1$, a criterion of this type for best approximation, as is known, does not exist in the general case). Here the function $(x - c)^{r-1}|x - c|^{1+\alpha}$ is considered, where r is a natural number and $-1 - 1/q < \alpha < 1$ ($\alpha \neq -1$), which reduces to $|x - c|^p$ when r is odd. The corresponding result for this more general case for uniform best approximations ($q = \infty$) is contained in the book of A. F. Timan ⁽⁶⁾ (see Sec. 7.2.2). It is also noted there (see p. 426) that for $\alpha = 0$, when $|c| < 1$, the exact order of decrease of the best approximations of this function by polynomials in the metric L_q on $[-1, 1]$ is equal to $1/n^{r+1/q}$.

An extension of the method of S. N. Bernstein ⁽³⁾ to the case of arbitrary $q \geq 1$ makes it possible to show that, as in the cases $q = \infty$, $q = 1$, the best approximation of the function $(x - c)^{r-1}|x - c|^{1+\alpha}$ in the metric L_q on $[-1, 1]$ for any $q \geq 1$ not only coincides in order, as $n \rightarrow \infty$, with the quantity $1/n^{r+\alpha+1/q}$, but is equal to it up to the precision of a certain constant factor.

Theorem 1. If $-1 < c < 1$, then, whatever the natural number r and $-1 - 1/q < \alpha < 1$ ($\alpha \neq -1$), for the best approximation $E_n[(x - c)^{r-1}|x - c|^{1+\alpha}; -1, 1]_{L_q}$ of the function $(x - c)^{r-1}|x - c|^{1+\alpha}$ by algebraic polynomials of degree not exceeding n in the metric of the space $L_q(-1, 1)$

($q \geq 1$), as $n \rightarrow \infty$ there exists the limit of the product $n^{r+\alpha+1/q} E_n[(x-c)^{r-1}|x-c|^{1+\alpha}; -1, 1]_{L_q}$, and

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{r+\alpha+1/q} E_n[(x-c)^{r-1}|x-c|^{1+\alpha}; -1, 1]_{L_q} &= \\ &= (\sqrt{1-c^2})^{r+\alpha+1/q} A_1(x^{r-1}|x|^{1+\alpha})_{L_q}, \end{aligned} \quad (1)$$

where $A_1[f(x)]_{L_q}$ is the best approximation of the function $f(x)$ by entire functions of degree not exceeding one in the metric of the space $L_q(-\infty, \infty)$.

Relation (1), when r is odd, for $q = \infty$ constitutes a theorem of S. N. Bernstein ⁽³⁾, and for $q = 1$ is a result of S. M. Nikol'skii ⁽⁵⁾. For arbitrary natural r and $q = \infty$, this asymptotic equality was proved in the book of A. F. Timan ⁽⁶⁾ (the case $c = 0$ for $q = \infty$ was studied by I. I. Ibragimov ⁽⁷⁾).

Theorem 1 shows how the metric L_q affects the asymptotics in the behavior of the sequence of best approximations by polynomials of the function under consideration. This influence appears still more clearly if one considers functions having on $[-1, 1]$ several algebraic

Theorem 2. Let

$$f(x) = \sum_{\nu=1}^h A_\nu (x - c_\nu)^{r-1} |x - c_\nu|^{1+\alpha},$$

where r is a natural number, $-1 - 1/q < \alpha < 1$ ($\alpha \neq -1$), $-1 < c_\nu < 1$. Then

$$\begin{aligned} E_n(f(x); -1, 1)_{L_q} &= \\ &= [1 + o(1)] \left\{ \sum_{\nu=1}^h |A_\nu|^q E_n^q[(x - c_\nu)^{r-1}|x - c_\nu|^{1+\alpha}; -1, 1]_{L_q} \right\}^{1/q}. \end{aligned} \quad (2)$$

From Theorems 1 and 2 there follows immediately

Theorem 3. If

$$f(x) = \sum_{\nu=1}^h A_\nu (x - c_\nu)^{r-1} |x - c_\nu|^{1+\alpha},$$

where r is a natural number, $-1 - 1/q < \alpha < 1$ ($\alpha \neq -1$), $-1 < c_\nu < 1$, then as $n \rightarrow \infty$ there exists the limit of the product $n^{r+\alpha+1/q} E_n(f(x); -1, 1)_{L_q}$, and

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{r+\alpha+1/q} E_n(f(x); -1, 1)_{L_q} = \\ & = \left\{ \sum_{\nu=1}^h |A_\nu|^q (\sqrt{1-c_\nu^2})^{(r+\alpha)q+1} \right\}^{1/q} A_1(x^{r-1}|x|^{1+\alpha})_{L_q}. \end{aligned} \quad (3)$$

Let us note that a proposition analogous to Theorem 3 is also valid for functions $f(x)$ of the form

$$f(x) = \sum_{\nu=1}^{\infty} A_\nu (x - c_\nu)^{r-1} |x - c_\nu|^{1+\alpha}, \quad \sum_{\nu=1}^{\infty} |A_\nu| < \infty.$$

Theorem 4. If

$$f(x) = \prod_{\nu=1}^h (x - c_k)^{r-1} |x - c_k|^{1+\alpha},$$

where r is a natural number; $-1 - 1/q < \alpha < 1$ ($\alpha \neq -1$), $-1 < c_k < 1$, $c_k \neq c_j$ ($k \neq j$), then as $n \rightarrow \infty$ there exists the limit of the product $n^{r+\alpha+1/q} E_n(f(x); -1, 1)_{L_q}$, and

$$\begin{aligned} & \lim_{n \rightarrow \infty} n^{r+\alpha+1/q} E_n(f(x); -1, 1)_{L_q} = \\ & = \left\{ \sum_{j=1}^h \prod_{\substack{k=1 \\ k \neq j}}^h |c_j - c_k|^{(r+\alpha)q} (\sqrt{1-c_j^2})^{(r+\alpha)q+1} \right\}^{1/q} A_1(x^{r-1}|x|^{1+\alpha})_{L_q}. \end{aligned}$$

The proof of the results presented is based on the general limit theorem of S. N. Bernstein (see ⁽⁸⁾, Theorem VII bis), as well as on a number of other auxiliary propositions relating to the case of arbitrary $q \geq 1$ and generalizing the corresponding theorems that are used in S. N. Bernstein's method for $q = \infty$.

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Note: Figure translations are in progress. See original paper for figures.

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