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Abstract

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ON SADDLE SURFACES BOUNDED BY A RECTIFIABLE CURVE

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The paper is devoted to the study of saddle surfaces in n-dimensional Euclidean space E^n $(n \geq 2)$. A saddle surface is a surface from which no hyperplane can cut off a cap. The exact definition of a saddle surface, which we shall use below, was given by Maksheĭn ([1]). In particular, regular surfaces of nonpositive Gaussian curvature are saddle surfaces.

It is known that such surfaces, as well as saddle polyhedra, possess an isoperimetric property: if S is the area of a simply connected domain on such a surface, and l is the length of the boundary of this domain, then the inequality $4\pi S - l^2 \leq 0$ holds.

Theorem 1. *If a simply connected saddle surface lies in the plane E^2 and is bounded by a contour of length l, then its area satisfies the inequality $4\pi S - l^2 \leq 0$.*

Further, since the area of a surface in E^n does not exceed the sum of the areas of its projections onto all two-dimensional coordinate planes ([2]), Theorem 1 implies the more general

Theorem 2. *If a simply connected saddle surface in E^n is bounded by a contour of length l, then its area S satisfies the inequality $A_n S - l^2 \leq 0$, where $A_n \geq 4\pi / C_n^2$. For $n > 2$ the estimate of the number A_n is not sharp.*

We briefly outline the plan of the proof of Theorem 1. First it is proved that Theorem 1 is valid for saddle polyhedra in E^2. Then, for an arbitrary saddle surface in E^2, a sequence of polyhedra is constructed which converges uniformly to this surface, and in such a way that, if the given saddle surface is bounded by a rectifiable contour of length l, then the lengths of the boundary contours of the approximating sequence of saddle polyhedra tend to l.*

The construction of such a sequence is based on a simple idea. Let an arbitrary saddle surface P in E^2 be given.

We construct a polyhedron $P_1(\varphi_1)$ close to it, defined by a mapping φ_1 of the square K into E^2 (in general, not saddle). If P_1 is not a saddle surface, then in K there exists a domain Q, whose boundary Q' is mapped onto some straight line in E^2, while the domain Q itself is not mapped onto this line. We correct the mapping φ_1, replacing it on Q by another mapping which already sends Q into the indicated straight line. Such a correction is called **cutting off caps**. It then turns out that the new mapping on Q can be defined so that the distance between P and the new surface remains small (formulas (*) and (**)). By successive application of the operation of cutting off caps, the polyhedron P_1 is transformed into a saddle one. The proof of Theorem 1 is then obtained by passing to the limit.

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* Hence, in particular, it follows that a saddle surface in E^2 bounded by a rectifiable contour carries a metric of nonpositive curvature in the sense of A. D. Aleksandrov.

For the proof of Theorem 1 we define the operation of cutting off humps. Let $F(u)$, $u \in K = [0,1;0,1]$, be a continuous function, R a real number, and let $Q_{F,R}$ be the union of all those components of the set $\{K, F(u) \neq R\}$ whose closures do not intersect the boundary of K.

Denote by $G_{F,R}$ the subset $G_F = \bigcup_R Q_{F,R}$ containing all components Q_F on whose boundary $F(u) = R$.

The operation δ is defined by the equality

$$\delta F(u) = \begin{cases} F(u), & \text{for } u \in K \setminus G_F, \\ R, & \text{for } u \in G_{F,R}. \end{cases}$$

The operation Δ consists in passing from the mapping $\varphi:\ x^i = \varphi^i(u)$, $i = 1,2$, to the mapping $\Delta\varphi:\ x^i = \delta\varphi^i(u)$, $i = 1,2$.

The operation $\Delta_\omega^{\varphi_0,\varepsilon}$. Let $\varepsilon > 0$ be fixed, let $\varphi_0:\ x^i = x_0^i(u)$, $i = 1,2$, be a saddle mapping of K into E^2, and let ω be a direction in E^2, characterized by the angular coefficient of a line with equation $x^2 = kx^1$ or $x^1 = kx^2$, so that $|k| \leq 1$. The operation $\Delta_\omega^{\varphi_0,\varepsilon}$ is defined on the space of mappings $\varphi:\ x^i = x^i(u)$, $i = 1,2$, $u \in K$, satisfying the conditions: a) $\Delta\varphi = \varphi$; b) $\rho(\varphi_0, \varphi) = \max_{i=1,2;\, u\in K}\{|x_0^i(u) - x^i(u)|\} \leq \varepsilon$, and consists in passing from the mapping φ to the mapping $\Delta_\omega^{\varphi_0,\varepsilon}\varphi:\ x^i = x_\omega^i(u)$, $i = 1,2$, $u \in K$, defined by the equalities $(*)$, if the direction ω makes with the axis x^1 an angle not greater than 45°, and by the equalities $(**)$ in the opposite case.

$$x_\omega^1(u) = \begin{cases} x^1(u), & \text{for } u \in K \setminus (\check{G}_{F_\omega} \cup \overset{*}{G}_{F_\omega}), \\ \dfrac{x_0^2(u) - R - \varepsilon}{k}, & \text{for } u \in \check{G}_{F_\omega R}, \\ \dfrac{x_0^2(u) - R + \varepsilon}{k}, & \text{for } u \in \overset{*}{G}_{F_\omega,R}; \end{cases}$$

$$x_\omega^2(u) = \begin{cases} x^2(u), & \text{for } u \in K \setminus G_{F_\omega}, \\ kx^1(u) + R, & \text{for } u \in G_{F_\omega} \setminus (\breve{G}_{F_\omega} \cup \overset{*}{G}_{F_\omega}), \\ x_0^2(u) - \varepsilon, & \text{for } u \in \breve{G}_{F_\omega}, \\ x_0^2(u) + \varepsilon, & \text{for } u \in \overset{*}{G}_{F_\omega}. \end{cases}$$

Here

$$F_\omega = x^2(u) - kx^1(u),$$

$$\breve{G}_{F_\omega, R} = \{G_{F_\omega, R},\ kx^1(u) + R - (x_0^2(u) - \varepsilon) < 0\}, \qquad \breve{G}_{F_\omega} = \bigcup_R \breve{G}_{F_\omega, R},$$

$$\overset{*}{G}_{F_\omega, R} = \{G_{F_\omega, R},\ kx^1(u) + R - (x_0^2(u) + \varepsilon) > 0\}, \qquad \overset{*}{G}_{F_\omega} = \bigcup_R \overset{*}{G}_{F_\omega, R}.$$

The formulas $(*, *)$ are obtained from $(*)$ by transposing the indices 1, 2.

The operation $\Delta\Delta_\omega^{\varphi_0, \varepsilon}$ is defined on the same class of mappings as $\Delta_\omega^{\varphi_0, \varepsilon}$, by the relation

$$\Delta\Delta_\omega^{\varphi_0, \varepsilon} \varphi = \Delta(\Delta_\omega^{\varphi_0, \varepsilon} \varphi).$$

By the methods of Makshane (^{1}), using the operations just defined, one proves

Lemma. An arbitrary saddle surface P in E^2 can be approximated by a sequence of saddle surfaces of class L^2, converging to P together with the lengths of the boundary contours.

The space $H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$. Let Φ be the space of all mappings of the square K into E^2 of class L^2. The space $H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$ is defined as a subspace of Φ by a number $\varepsilon > 0$, a fixed finite

by a sequence Ω_n of n directions in E^2 distinct from the directions of the axes x^1, x^2, a saddle mapping φ_0, and a mapping φ such that $\rho(\varphi_0, \varphi) \leqslant \varepsilon$.

A mapping ψ, by definition, belongs to $H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$ if at least one of the following conditions is fulfilled: a) $\psi = \Delta\varphi$; b) $\psi = \Delta\Delta_\omega^{\varphi_0, \varepsilon} \psi'$, where $\psi' \in H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$, and $\omega \in \Omega_n$; c) $\psi = \lim_{k \to \infty} \psi'_k$, where $\psi'_k \in H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$.

Properties of the space $H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$:

1) if $\psi \in H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$, then $\rho(\varphi_0, \psi) \leqslant \varepsilon$;

2) if $\psi \in H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$, then $\Delta\psi = \psi$;

3) if $\Delta\Delta_\omega^{\varphi_0,\varepsilon}\psi \neq \psi$, then $L[\Delta\Delta_\omega^{\varphi_0,\varepsilon}\psi] < L[\psi]$, where $L[\psi]$ is the area of the surface defined by the mapping ψ;
4) in $H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$ there exists a mapping $\psi_0\{\varphi_0, \varphi, \Omega_n\}$ having the least area in it; moreover, for arbitrary $\omega \in \Omega_n$,

$$\Delta\Delta_\omega^{\varphi_0,\varepsilon}\psi_0 = \psi_0.$$

Properties 1), 2), 3) are proved in an elementary way; property 4) is proved by the method of [1], using properties 1), 2), 3) and formulas (*) and (**).

Proof of Theorem 1. Let $P_0(\varphi_0 : x^i = \varphi_0^i(u),\ i = 1, 2,\ u \in K)$ be a saddle surface of class L^2, bounded by a contour of length l. Fix $\varepsilon > 0$. Construct a polyhedron $P(\varphi : x^i = \varphi^i(u),\ i = 1, 2,\ u \in K)$ satisfying the following conditions:

1. $\rho(\varphi_0, \varphi) \leqslant \varepsilon$.
2. The mapping φ is nondegenerate on none of the triangles of the corresponding simplicial subdivision of K.
3. The length of the boundary contour of the surface P is close to l.

Let now $\Omega(\omega_1, \omega_2, \dots, \omega_n, \dots)$ be a countable everywhere dense set of directions in the plane E^2, distinct from the directions of the axes x^1, x^2, and let Ω_n be the first n members of Ω.

Construct a sequence of surfaces

$$P_1(\varphi_1),\ P_2(\varphi_2), \dots,\ P_n(\varphi_n), \dots \tag{1}$$

according to the following rule: φ_1 coincides with φ, and φ_n $(n > 1)$ is the mapping $\psi_0\{\varphi_0, \varphi_{n-1}, \Omega_{n-1}\}$, belonging to $H\{\varphi_0, \varphi_{n-1}, \Omega_{n-1}, \varepsilon\}$. By the methods of [1], proceeding from the properties of the space $H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$, one can prove that from the sequence (1) one can extract a subsequence converging to some surface $P^*(\varphi^*)$ uniformly in the Fréchet sense. From properties 1), 4) of the space $H\{\varphi_0, \varphi, \Omega_n, \varepsilon\}$ and from the choice of the sequence Ω it follows that P^* is a saddle surface close to P_0.

We shall prove that P^* is a polyhedron. The latter is based on the following considerations. From the definition of the operations of cutting off humps it follows that P^* consists of a finite number of plane convex sets (each of these sets represents a part of a face of the polyhedron P which is not cut off in the construction of the sequence (1)), and the intersection of two such sets on the saddle surface consists of no more than one component, which can only be a segment or a point. In conclusion of the proof we note that for a polyhedron Theorem 1 is obvious. The idea of using projections of the surface onto a plane in the proof of Theorem 2 belongs to Yu. G. Reshetnyak.

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CITED LITERATURE

1. E. J. McShane, Trans. Am. Math. Soc., **35**, 716 (1933).
2. L. Cesari, Ann. Math. Studies, No. 35, Princeton (1956).

Note: Figure translations are in progress. See original paper for figures.