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Abstract

Full Text

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On Local Best Approximation of Functions by Polynomials

(Presented by Academician S. N. Bernstein, 27 X 1964)

Mathematics

1. In the present note we give some new results on the connection between the properties of a function and the behavior of its local best approximations. The first works in this direction were carried out by S. N. Bernstein ⁽¹⁾ and D. A. Raikov ⁽⁴⁾; in the time that has passed since then, comparatively few works have been done in which the above-mentioned connection was investigated explicitly (we know only the work ⁽²⁾); however, some results in the theory of approximations (as a whole) and in the theory of functions of a real variable can be assigned to the domain under consideration (see, for example, ^(3,5,6)). It seems to us, nevertheless, that the domain of investigation under consideration, which is a borderline one between approximation theory (as a whole) and the theory of functions of a real variable, is of undoubted interest.
2. Let $B(I_0)$ be one of the functional spaces $L^p(I_0)$, $1 \leq p \leq \infty$, $C(I_0)$, or $M(I_0)^*$, where I_0 is an interval of the real axis, not necessarily finite, and $\|f; I\|$, where $I \subset I_0$, is the corresponding local norm, i.e.

$$\|f; I\| = \left(\int_I |f(x)|^p dx \right)^{1/p},$$

if $f \in L^p(I_0)$, and

$$\|f; I\| = \sup_{x \in I} |f(x)|,$$

if $f \in C(I_0)$ or $f \in M(I_0)$. Everywhere in what follows I denotes a finite interval from I_0 , and $|I|$ its length. Denote by $E_r(f; I)$ the quantity $\inf \|f - p; I\|$, where the infimum is taken over all polynomials p of degree $\leq r$; a polynomial on which the infimum is attained will be called a polynomial of best approximation for f on I . Sometimes, instead of $E_r(f; I)$, it is more convenient to consider the quantity $E_T(f; I)$, defined as follows. Let \mathfrak{M}_r denote the class of linear operators acting from $B(0, 1)$ into the space of polynomials of degree r and leaving the polynomials of this space fixed. Let K_I be the isometric operator acting from $B(I)$ into $B(0, 1)$ by the formula

$$K_I f = |I|^\gamma f(|I|x + a),$$

where a is the left endpoint of I ; here and everywhere below γ is equal to $1/p$ for $L^p(I_0)$ and to 0 for the other spaces; then, if we put $T_I = K_I^{-1}TK_I$, then $E_T(f; I) = \|f - T_I f; I\|$. If $T \in \mathfrak{M}_r$, then, as is known,

$$E_r(f; I) \leq E_T(f; I) \leq (1 + \|T\|)E_r(f; I), \quad (1)$$

therefore most of the results presented remain valid when $E_r(f; I)$ is replaced by $E_T(f; I)$.

3. By Ω_r we denote the class of step functions $\rho(t)$ with a finite number of jumps at the points $\{a_j\}_{j=1}^N$ such that $a_1 = 0$, and

$$\int_{-\infty}^{\infty} t^i d\rho(t) = 0$$

for $0 \leq i < r$ and $\neq 0$ for $i = r$. If $\rho(t) \in \Omega_r$, then by the r -oscillation of a function $f(x) \in B(I_0)$ on $I \subset I_0$ we shall call the functional

$$\omega_\rho(f; I) = \sup_h \left\| \int_{-\infty}^{\infty} f(x + th) d\rho(t); I_{\rho, h} \right\|.$$

* $M(I_0)$ is the space of bounded (not necessarily measurable) functions on I_0 .

Here $I_{\rho, h} = \{x : x + \alpha_{jh} \in I, j = 1, \dots, N\}$. Let us note that if $\rho(t) \in \Omega_1$, then in the uniform metric $\omega_\rho(f; I) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x)$, which also explains the name $\omega_\rho(f; I)$. For the function $\rho_0(t) = \sum_{j \leq t} (-1)^{r-j} \binom{r}{j}$ ($\in \Omega_r$) we shall write $\omega_r(f; I)$ instead of $\omega_\rho(f; I)$.* The following theorem, important for what follows, indicates the connection between $\omega_\rho(f; I)$ and $E_{r-1}(f; I)$.

Theorem 1. If $f(x) \in B(I_0)$ and is measurable, then for every $\rho(t) \in \Omega_r$ there exist constants $A > 0$ and $B > 0$, depending only on $\rho(t)$, such that

$$A\omega_\rho(f; I) \leq E_{r-1}(f; I) \leq B\omega_\rho(f; I). \quad (2)$$

For the case $\omega_r(f; I)$ and the space $M(I_0)$ the theorem was first proved by H. Whitney [5] (without the assumption of measurability). We note that our method of proof makes it possible, in the uniform metric, to prove (2) assuming only that f is measurable on I , which answers a question of H. Whitney concerning $\omega_r(f; I)$ (see [5]).

Let us record consequences of (2).

Corollary 1. If $\rho_1(t)$ and $\rho_2(t)$ belong to Ω_r , then there exist constants $A > 0$, $B > 0$, depending only on $\rho_1(t)$ and $\rho_2(t)$, such that

$$A\omega_{\rho_1}(f; I) \leq \omega_{\rho_2}(f; I) \leq B\omega_{\rho_1}(f; I). \quad (3)$$

It follows from (3) that it is enough to use one of the r -oscillations. We shall use $\omega_r(f; I)$.

Corollary 2. In order that $f(x) \in B(I_0)$ be equivalent on $I \subset I_0$ (in the case $M(I_0)$, equal) to a polynomial of degree $\leq r - 1$, it is necessary and sufficient that $\omega_r(f; I) = 0$.

4. In the present section we shall establish a connection between the existence of derivatives of the function $f(x)$ (on the whole interval or at a point) and the behavior of local best approximations. We shall use the following definitions of derivatives (the first of them is due to Vallée-Poussin and Zygmund).

The function $f(x) \in B(I_0)$ has at the point $x_0 \in I_0$ a strong derivative of order r (notation $D^r f(x_0)$) if there exists a polynomial $p(x) = \sum_{k=0}^r a_{kx}^k$ such that

$$\|f - p; I_h(x_0)\| = o(h^r)$$

as $h \rightarrow +0$. Here $I_h(x_0) = [x_0 - h, x_0 + h] \cap I_0$; moreover, by definition, $D^r f(x_0) = r!a_r$. The function $f(x) \in B(I_0)$ has at the point $x_0 \in I_0$ a weak derivative of order r (notation $D_{sl}^r f(x_0)$) if

$$\|h^{-r} \Delta^r(h)f - a; I_h(x_0)\| = o(h^r)$$

as $h \rightarrow +0$; here $a = D_{sl}^r f(x_0)$. Let us note that if $D^r f(x_0)$ exists at the point x_0 , then $D_{sl}^r f(x_0)$ also exists. The converse is false; however, if $D_{sl}^r f(x_0)$ exists for all $x_0 \in I$ and the relation

$$\|h^{-r} \Delta^r(h)f - D_{sl}^r f(x_0); I_h(x_0)\| = o(h^r)$$

holds uniformly with respect to $x_0 \in I$, then $D^r f(x)$ exists everywhere on I . Let us also note that for $r = 1$ and the space $M(I_0)$ both definitions coincide with the classical definition of derivative. Finally, by the r -th derivative of a function $f(x) \in B(I_0)$ on I_0 (notation $f^{(r)}(x)$) we shall mean the Sobolev r -th derivative if $B(I_0) = L^p(I_0)$, and the classical r -th derivative in the other cases.

Theorem 2. Let $f(x) \in B(I_0)$, $x_0 \in I_0$, and let $p(x; I)$ denote a polynomial of degree $\leq s$, where $s \geq r$, least deviating from $f(x)$ on I . Then

* Let us note that in this case $\int_{-\infty}^{\infty} f(x+th) d\rho_0(t) = \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} f(x+jh) = \Delta^r(h)f(x)$.

A. F. Timan drew my attention to this generalization of the difference operation conditions

$$\lim_{I \rightarrow x_0} \frac{E_r(f; I)}{|I|^{r+\gamma}} = 0, \quad \lim_{I \rightarrow x_0} \frac{\|f^{(r)}(x; I) - a; I\|}{|I|^\gamma} = 0, \quad (4)$$

where a is some constant, are necessary for the existence of $D^r f(x_0)$ (with $a = D^r f(x_0)$) and sufficient for the existence of $D_{\text{sl}}^r f(x_0)$ (in this case $D_{\text{sl}}^r f(x_0) = a$).

Remark 1. $\lim_{I \rightarrow x_0}$ denotes the limit over the partially ordered set of intervals I containing x_0 . Let us note that if the limits in (4) are taken along some decreasing sequence of intervals $\{I_j\}$ containing x_0 and such that $|I_{j+1}|/|I_j| \geq \lambda > 0$, $j = 1, 2, \dots$, then Theorem 2 remains valid.

Remark 2. It can be shown by examples that the fulfillment of only one of the conditions (4) is insufficient for the existence of $D_{\text{sl}}^r f(x_0)$.

The following theorem is a generalization of a result of S. N. Bernstein ⁽¹⁾.

Theorem 3. Let $f(x) \in B(I_0)$ and let $D^r f(x_0)$ exist ($x_0 \in I_0$). Then

$$\lim_{I \rightarrow x_0} \frac{E_{r-1}(f; I)}{|I|^{r+\gamma}} = c_r |D^r f(x_0)|. \quad (5)$$

Here

$$c_r = \frac{1}{r!} E_{r-1}(x^r; [0, 1])$$

(in the spaces $M(I_0), C(I_0), L^1(I_0)$, and $L^\infty(I_0)$, c_r , as is known, is equal to $1/2^{r-1} r!$; in the space $L^2(I_0)$ it is equal to $(r!/(2r)!) \times (1/\sqrt{2r+1})$; for other $B(I_0)$ the exact value of c_r is unknown).

Remark. For the space $C(I_0)$, (5) is also true under the assumption of the existence of $D_{\text{sl}}^r f(x_0)$.

In the following theorems, $\varphi(t)$ denotes a monotonically increasing function together with t , such that for some constant $C > 0$,

$$\varphi(2t) \leq C\varphi(t)$$

for all t .

Theorem 4. If $E_r(f; I) \leq \varphi(|I|)$ for all $I \subset I_0$ and the integral

$$\int_0^{|I|} \frac{\varphi(t)}{t^{s+\gamma+1}} dt$$

converges for some natural $s \leq r$, then $f^{(s)}(x)$ exists and

$$E_{r-s}(f^{(s)}; I) = O\left(|I|^\gamma \int_0^{|I|} \frac{\varphi(t)}{t^{s+\gamma+1}} dt\right), \quad I \subset I_0. \quad (6)$$

Theorem 5. If $E_r(f; I) \leq \varphi(|I|)$ for all $I \subset I_0$, then for an integer s ($0 \leq s < r$) we have

$$E_s(f; I) = O \left(|I|^{s+\gamma+1} \int_{|I|}^{|I_0|} \frac{\varphi(t)}{t^{s+\gamma+2}} dt \right), \quad I \subset I_0. \quad (7)$$

Theorem 6. Let $f(x) \in L^p(I_0)$ and $E_r(f; I)_{L^p} \leq \varphi(I)$ for all $I \subset I_0$. Then, if the integral

$$\int_0^1 \frac{\varphi(t)}{t^{1+\gamma}} dt$$

converges, then $f(x) \in L^\infty(I_0)$ and

$$E_r(f; I)_{L^\infty} = O \left(\int_0^{|I|} \frac{\varphi(t)}{t^{1+\gamma}} dt \right). \quad (8)$$

Among the numerous consequences of Theorems 4-6 we mention only the following:

Corollary. The condition $E_r(f; I) = O(|I|^{s+\gamma+\alpha})$, $I \subset I_0$, for $0 < \alpha < 1$ is necessary and sufficient in order that $f^{(s)} \in L^\infty(I_0)$ and

$$\text{vrai sup}_{x \in I_0} |\Delta^1(h)f^{(s)}(x)| = O(h^\alpha),$$

and for $\alpha = 0$, in order that $f^{(s-1)} \in L^\infty(I_0)$ and

$$\text{vrai sup}_{x \in I_0} |\Delta^2(h)f^{s-1}(x)| = O(h).$$

5. In paper (1), S. N. Bernstein proved a criterion for the existence, for a function in $C(I_0)$, of a continuous r -th derivative (in terms of local approximations). In the present section we shall give criteria for the membership of $f^{(r)}(x)$ in $L^p(I_0)$ or $V(I_0)$ ($f \in V(I_0)$, if f is equivalent to a function of bounded variation on I_0). For brevity put

$$R_{p,q}^{(r)}(f; \{I_s\}) = \left\{ \sum_s \frac{E_{r-1}^p(f; I_s)_{L^q} L^q}{|I_s|^{rp+p/q-1}} \right\}^{1/p},$$

where $f(x) \in L^q(I_0)$, and $\{I_s\}$ is a system of nonoverlapping intervals from I_0 .

Theorem 7. *The condition*

$$\sup_{\{I_s\}} R_{p,q}^{(r)}(f; \{I_s\}) < \infty$$

for $1 < p \leq q \leq \infty$ is necessary and sufficient for the existence of $f^{(r)}(x) \in L^p(I_0)$, and for $p = 1 \leq q \leq \infty$ —for the existence of $f^{(r-1)}(x) \in V(I_0)$.

Remark. For $p = q = \infty$ the result can be sharpened as follows: the condition

$$E_{r-1}(f; I)_{L^\infty} \leq M|I|^r/r!2^{2r-1}$$

for all $I \subset I_0$ is necessary and sufficient for the existence of $f^{(r)}(x) \in L^\infty(I_0)$, with

$$\|f^{(r)}; I_0\| \leq M$$

(see (2)).

Theorem 8. *In order that $f^{(r)}(x) \in L^1(I_0)$ exist, it is necessary and sufficient that for every $\varepsilon > 0$ there be a $\delta > 0$ such that whenever*

$$\sum_s |I_s| < \delta,$$

where $\{I_s\}$ is a system of nonoverlapping intervals from I_0 , one has

$$R_{1,q}^{(r)}(f; \{I_s\}) < \varepsilon.$$

6. Most of the results stated above are also valid for functions of n variables. In this case, to characterize “mixed” properties of a function (such as the existence of a mixed derivative), one uses approximation by quasipolynomials, i.e., by functions of the form

$$\sum_{i=1}^n \sum_{s=0}^{k_i} f_s(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) x_i^s,$$

while to characterize properties of the function as a whole (such as the existence of a total differential), one uses approximation by algebraic polynomials.

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