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Abstract

Full Text

MATHEMATICS

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DIVISION AND FACTORIZATION OF LAURENT SERIES

(Presented by Academician M. A. Lavrent' ev, 20 XI 1964)

Let a Laurent series be given

$$f(z) = \sum_{m=-\infty}^{\infty} a_m z^m, \tag{1}$$

convergent in the annulus

$$r < |z| < R. \tag{2}$$

Consider a system of annuli

$$\{r_{p_i} < |z| < r_{p_{i+1}}\}. \tag{3}$$

Each of the annuli

$$r_{p_i} < |z| < r_{p_{i+1}} \tag{4}$$

of this system satisfies the conditions: a) $r \le r_{p_i} < |z| < r_{p_{i+1}} \le R$; b) the function $f(z)$ has no zeros in (4); c) the function $f(z)$ has at least one zero on each of the circles $|z| = r_{p_i}$ and $|z| = r_{p_{i+1}}$ (where $r_{p_i} \ne r$, $r_{p_{i+1}} \ne R$).

To each annulus (4) we assign an integer (number) p_i, equal to

$$p_i = I(\rho) = \int_{|z|=\rho} \frac{f'(z)}{f(z)}\, dz, \qquad r_{p_i} < \rho < r_{p_{i+1}}. \tag{5}$$

In (^{1}) this number p_i is called the principal index of the series (1). Obviously, the difference $p_i - p_{i-1} = I(r_{p_i} + 0) - I(r_{p_i} - 0)$ is equal to the number of zeros of $f(z)$ on the circle $|z| = r_{p_i}$.

In each of the annuli (4) the quotient $\varphi(z) : f(z)$ is uniquely expanded into a Laurent series

$$\frac{\varphi(z)}{f(z)} = \sum_{m=-\infty}^{\infty} d_{m,i} z^m, \tag{6}$$

where $\varphi(z)$ is an arbitrary Laurent series convergent in (2):

$$\varphi(z) = \sum_{m=-\infty}^{\infty} c_m z^m. \tag{7}$$

Even for polynomials, the numerical determination of the coefficients of the series (6) in the annulus (4) is usually connected with great difficulties.

In this note a method is proposed for dividing Laurent series, by means of which one can numerically determine the coefficients $d_{m,i}$ in any of the annuli (4) of the system; a new algorithm is also proposed for the factorization of Laurent series (in particular, of power series and polynomials). For the series (1), the system of annuli (3) may consist of a countable or finite number of annuli (4); in particular, it consists of a single annulus (2) if $f(z)$ has no zeros in the annulus of convergence.

Lemma. *The function $f(z)$, given by the Laurent series (1), in each annulus (4) can be represented in the form*

$$f(z) = z^{p_i} q_{p_i} \exp \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{b_{m,i}}{m} z^m,$$

where in (6) we have put $\varphi(z) \equiv f'(z)$ and

$$\frac{f'(z)}{f(z)} = \sum_{m=-\infty}^{\infty} b_{m,i} z^{m-1}; \tag{8}$$

p_i is the number of the annulus (4); p_i is determined by formulas (5); q_{p_i} is a certain constant depending on p_i.

Integrating the left- and right-hand sides of (8) from z_0 to z ($r_{p_i} < |z| < r_{p_i+1}$), we obtain:

$$\ln \frac{f(z)}{z^{p_i}} - \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{b_{m,i}}{m} z^m = \ln \frac{f(z_0)}{z_0^{p_i}} - \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{b_{m,i}}{m} z_0^m.$$

It follows immediately from this that the assertion of the lemma holds, with

$$q_{p_i} = \frac{f(z_0)}{z_0^{p_i}} \exp\left(- \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{b_{m,i}}{m} z_0^m \right).$$

Theorem 1. The function $f(z)$, given by the series (1), can be represented in the annulus (2) in the form

$$f(z) = z^{p_i} q_{p_i} f_{p_i}^{-}(z) f_{p_i}^{+}(z), \tag{9}$$

where $f_{p_i}^{-}(z)$ is a function holomorphic in the circle $|z| > r$, $f_{p_i}^{-}(\infty) = 1$; $f_{p_i}^{+}(z)$ is holomorphic in $|z| < R$, $f_{p_i}^{+}(0) = 1$; p_i is the principal index of $f(z)$. For a given p_i, the representation (9) has the uniqueness property.

Let us now consider the following functions:

$$f_k(z) = \prod_{j=0}^{k-1} f(\omega_j^{(k)} z) = \sum_{m=-\infty}^{\infty} a_m^{(k)} z^{km}, \tag{10}$$

$$F_k(z) = \varphi(z) \prod_{j=1}^{k-1} f(\omega_j^{(k)} z) = \sum_{m=-\infty}^{\infty} c_m^{(k)} z^m, \tag{11}$$

where $\omega_j^{(k)} = \exp \frac{2\pi i}{k} j$; $j = 0, 1, 2, \ldots, k-1$; $k = 1, 2, \ldots$

The function (11) can be represented in the form

$$F_k(z) = (-1)^{p_i(k-1)} q_{p_i}^k z^{kp_i} \frac{\varphi(z)}{f(z)} \exp \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{b_{km,i}}{m} z^{km}. \tag{12}$$

In (?) it is shown that, in the annulus $r_{p_i} + \varepsilon < |z| < r_{p_i+1} - \varepsilon$, the sequence of functions

$$U(z) = \exp \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{b_{km,i}}{m} z^m$$

tends uniformly to one as $k \to \infty$, however small $\varepsilon > 0$ may be. Therefore (12) may be written as

$$\frac{\varphi(z)}{f(z)} = (-1)^{p_i(k-1)} \lim_{k\to\infty} \frac{F_k(z)}{q_{p_i}^k z^{kp_i}}. \tag{13}$$

Putting $\varphi(z) \equiv f'(z)$ in (13), we obtain the coefficients of the expansion (8), and hence can find the factors

$$f_{p_i}^{-}(z) = \exp \sum_{m=-\infty}^{-1} \frac{b_{m,i}}{m} z^m, \qquad f_{p_i}^{+}(z) = \exp \sum_{m=1}^{\infty} \frac{b_{mi}}{m} z^{-m}. \tag{14}$$

Introduce the functions

$$M_k(z) = \prod_{j=1}^{k} f(\tau_j^{(k)} z), \qquad N_k(z) = \prod_{j=1}^{k} f[(\tau_j^{(k)})^{-1} z], \tag{15}$$

where $\tau_j^{(k)}$ are the roots of the polynomial $\psi(z) = z^k - z^{k-1} + 1$. For the roots of the polynomial $\psi(z)$ it is not difficult to establish the inequalities

$$(1/2)^{1/k-1} < |\tau_j^{(k)}| < k^{1/k}, \qquad (1/k)^{1/k} < |(\tau_j^{(k)})^{-1}| < 2^{1/k-1}, \qquad j = 1, 2, \dots, k.$$

Using the same methods as in the derivation of (13), we obtain

$$f_{p_i}^{-}(z) = \lim_{k\to\infty} \frac{N_k(z)}{[q_{p_i}(-z)^{p_i}]^k}, \qquad f_{p_i}^{+}(z) = \lim_{k\to\infty} \frac{M_k(z)}{[q_{p_i}(-z)^{p_i}]^k}, \tag{16}$$

where

$$S_m^{(k)} = \sum_{j=1}^{k} (\tau_j^{(k)})^{-1} \begin{cases} 0, & \text{if } -k+1 \le m \le -1, \\ 1, & \text{if } 1 \le m \le k+1. \end{cases}$$

The same formula also holds for $N_k(z)$ (instead of $S_m^{(k)}$ there is the factor $S_{-m}^{(k)}$).

The processes (13) and (16) converge uniformly inside the annulus (4). For a variable z satisfying the condition $r_{p_i} < \rho_i < |z| < \rho_{i+1} < r_{p_{i+1}}$, convergence in (13) is characterized by the relation

$$\left| \frac{F_k(z)}{q_{p_i}^k (-z)^{kp_i} (-1)^{p_i}} - \frac{\varphi(z)}{f(z)} \right| = o\left(\left| \frac{r_{p_i}}{\rho_i} \right|^k + \left| \frac{\rho_{i+1}}{r_{p_{i+1}}} \right|^k \right).$$

The same convergence also holds in (16).

For computing the coefficients of the series (10), recurrence formulas were proposed in (3,4) for $k \ge 2$. Analogous formulas for $k \ge 2$ can also be proposed for determining the coefficients of the series (11). These formulas will be simplest when k takes the discrete values $1, 2, 4, 8, 16, \dots$,

$$f_{2k}(z) = f_k(z) f_k[(-1)^{1/k} z], \qquad F_{2k}(z) = F_k(z) f_k[(-1)^{1/k} z],$$

where

$$a_m^{(2k)} = (-1)^m \left[(a_m^{(k)})^2 + 2 \sum_{j=1}^{\infty} (-1)^j a_{m-j}^{(k)} a_{m+j}^{(k)} \right],$$

$$c_m^{(2k)} = \sum_{j=-\infty}^{\infty} (-1)^j a_j^{(k)} c_{m-kj}^{(k)}, \tag{17}$$

$$a_m^{(1)} = a_m, \qquad c_m^{(1)} = c_m; \qquad m = 0, \pm 1, \pm 2, \dots; \qquad k = 1, 2, \dots .$$

From (6), (13), and (17) we can write

$$|d_{m,i} - c_{m+kp_i}^{(k)} / a_{p_i}^{(k)}| = o(\rho^k), \qquad r_{p_i} / r_{p_{i+1}} < \rho < 1.$$

If $r_{p_i} \neq r$ and $r_{p_{i+1}} \neq R$, then $o(\rho^k)$ in the last equality may be replaced by $o(|r_{p_i}/r_{p_{i+1}}|^k)$.

The coefficients of the series (11) can also be computed by the formulas

$$F_k^{(m)}(z) = \frac{\varphi(z)}{z^m} \prod_{j=1}^{k-1} f(\omega_j^{(k)} z), \qquad Q_k^{(m)}(z) = \sum_{j=0}^{k-1} F_k^{(m)}(\omega_j^{(k)} z),$$

$$Q_{2k}^{(m)}(z) = Q_k^{(m)}(z) f_k[(-1)^{1/k} z] + Q_k^{(m)}[(-1)^{1/k} z] f_k(z),$$

where

$$c_{m+kp}^{(2k)} = (-1)^p \left[a_p^{(k)} c_{m+kp}^{(k)} + \sum_{j=1}^{\infty} (-1)^j \left(a_{p-j}^{(k)} c_{m+(p+j)k}^{(k)} + a_{p+j}^{(k)} c_{m+(p-j)k}^{(k)} \right) \right]. \tag{18}$$

By formulas (18) it is convenient to compute the coefficients for fixed m and variable p.

Formulas (17) and (18) can also be used for computing the coefficients of the factors $f_{p_i}^-, f_{p_i}^+$ in (9). For this it is necessary to put $c_m^{(1)} = (m+1) a_{m+1}$ in (7) and (11) to compute, by the method described above, the coefficients of the logarithmic derivative (8) in the annulus (4), using formulas (13) and (17). Then, by the formulas

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$$t^+_{m,i} = \frac{1}{m}\sum_{j=1}^{m} t^+_{m-j,i} b_{j,i}, \qquad t^+_{0,i} = 1,$$

$$t^-_{m,i} = -\frac{1}{m}\sum_{j=1}^{m} t^-_{m-j,i} b_{j,i}, \qquad t^-_{0,i} = 1, \tag{19}$$

$$m = 0, 1, 2, \dots,$$

one can find the expansion (14)

$$f^+_{p_i}(z) = \exp\sum_{m=1}^{\infty}\frac{b_{m,i}}{m} z^m = \sum_{m=1}^{\infty} t^+_{m,i} z^m,$$

$$f^-_{p_i}(z) = \exp\sum_{m=-\infty}^{-1}\frac{b_{m,i}}{m} z^m = \sum_{m=-\infty}^{-1} t^-_{m,i} z^m.$$

In formula (18) it is important that p be a principal index. Necessary and sufficient conditions for a principal index were given in (^{1,5}).

Suppose that in an arbitrarily chosen annulus

$$r < r_1 < |z| < R_1 < R \tag{20}$$

the function $f(z)$ has zeros $z_1, z_2, \dots, z_n$.

Theorem 2. The function $f(z)$ is represented in the annulus (2) in the form

$$f(z) = z^{p_i} q_{p_i} \prod_{i=1}^{n}\left(1 - \frac{z}{z_i}\right) f^+_{p_i+n} f^-_{p_i} = z^{p_i+n} q_{p_i+n} \prod_{j=1}^{n}\left(1 - \frac{z_i}{z}\right) f^-_{p_i} f^+_{p_i+n}, \tag{21}$$

where $p_i = I(r_1 + 0)$, $p_{i+n} = I(R_1 - 0) = p_k$. The polynomial in formula (21) with roots from the annulus (20) can be determined by the formulas

$$\prod_{j=1}^{n}\left(1 - \frac{z}{z_i}\right) = f^+_{p_i}(z) : f^+_{p_i+n}(z), \qquad \prod_{j=1}^{n}\left(1 - \frac{z_i}{z}\right) = f^-_{p_i+n}(z) : f^-_{p_i}(z),$$

or one may use Newton’ s formulas, computing the power sums of the zeros of the polynomial by the formulas

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$$S_{-m} = \sum_{j=1}^{n} z_j^{-m} = b_{m,k} - b_{m,i}, \qquad m = \pm 1, \pm 2, \dots .$$

In (^{2}) it is shown that the principal indices p_i can be computed in the process of transforming the given series by formulas (10), rather than using formulas (5) for this purpose.

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REFERENCES

[1] I. V. Viten' ko, A. N. Kostovskii, DAN, 155, No. 4 (1964).
[2] I. V. Viten' ko, Dop. AN UkrSSR, 9, No. 1 (1963).
[3] I. V. Viten' ko, O. M. Kostovs' kyi, *Theoretical and Applied Mathematics*, Publishing House of Lviv University, issue 2, 31 (1963).
[4] A. N. Kostovskii, *Journal of Computational Mathematics and Mathematical Physics*, 1, No. 2, 345 (1961).
[5] A. N. Kostovskii, DAN, 147, No. 2 (1962).

Note: Figure translations are in progress. See original paper for figures.

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