



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

T. A. TIMAN

1965

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.88931>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1965. Volume 162, No. 5

MATHEMATICS

T. A. TIMAN

ON THE EXTENSION OF HÖLDER-CONTINUOUS FUNCTIONS

(Presented by Academician S. N. Bernstein on 12 XII 1964)

Let K be an arbitrary metric space, where the distance between any pair of elements $x \in R$, $y \in R$ is determined by a certain nonnegative real function $\rho(x, y)$ with the properties $\rho(x, y) = 0$ if and only if $x = y$, and

$$\rho(x, y) \leq \rho(x, z) + \rho(y, z).$$

A real function $f(x)$ defined on R is called **Hölder-continuous** if, for some positive $\alpha \leq 1$ and some $C < \infty$, for any two elements $x_1, x_2 \in R$ for which $\rho(x_1, x_2) \leq 1$, the inequality

$$|f(x_1) - f(x_2)| \leq C\{\rho(x_1, x_2)\}^\alpha \quad (1)$$

holds.

Numerous problems of analysis are connected with the concept of Hölder continuity. In this note we wish to point out one addition to a known general theorem on the extension of continuous functions (see ⁽¹⁾, Chap. VI, § 13), which refines this theorem for the case when the functions under consideration are Hölder-continuous.

Theorem. *Whatever closed set of the metric space R is taken, and whatever bounded real Hölder-continuous function $\varphi(x)$ is defined on it, it can be extended to the whole space R without increasing the maximum of its modulus, in such a way that the new function $f(x)$ remains Hölder-continuous.*

Let A and B be two closed subsets of the space R , the distance d between them being positive. For any two real numbers a and b ($a < b$) and for any positive $\alpha \leq 1$, there exists a function $F(x)$, defined in the whole space and such that $a \leq F(x) \leq b$, $F(x) = a$ for $x \in A$, $F(x) = b$ for $x \in B$, and

$$|F(x_1) - F(x_2)| \leq (b - a)\{\rho(x_1, x_2)/d\}^\alpha. \quad (2)$$

As in the consideration of all continuous functions, one constructs a sequence of functions $\varphi_n(x)$, defined on the set P , by the recurrence formula

$$\varphi_n(x) = \varphi_{n-1}(x) - f_{n-1}(x) \quad (n \geq 1),$$

where $\varphi_0(x) = \varphi(x)$; $f_n(x)$ is the function indicated above, constructed for the sets A_n and B_n , for which, respectively, $\varphi_n(x) \geq \mu_n/3$ and $\varphi_n(x) \leq -\mu_n/3$, with

$$\mu_n = \max_{x \in P} |\varphi_n(x)|$$

and $a = -\mu_n/3$, $b = \mu_n/3$. Passing to the consideration of Hölder-continuous functions, we introduce a sequence of distances d_n between the sets A_n and B_n , and establish the following proposition, which plays the principal role here.

Lemma. *If the function $\varphi(x)$, for some positive $\alpha \leq 1$ and some positive C_0 , satisfies, for distinct pairs of points $x_1 \in P$, $x_2 \in P$, the condition*

$$|\varphi(x_1) - \varphi(x_2)| \leq C_0 \{\rho(x_1, x_2)\}^\alpha, \quad (3)$$

then, whatever nonnegative integer value of n is taken, it is true inequality

$$C_0 d_n^\alpha \geq \frac{1}{2^{n-1}} \frac{\mu_n}{3}. \quad (4)$$

Inequality (4) is connected with the inequality

$$|\varphi_n(x_1) - \varphi_n(x_2)| \leq 2^n C_0 \{\rho(x_1, x_2)\}^\alpha \quad (5)$$

and, together with it, can be obtained by applying double induction.

The lemma given above makes it possible to show that the function $f(x) = \sum_{n=0}^{\infty} f_n(x)$, which is an extension of the function $\varphi(x)$, is Hölder continuous on the whole space R , provided that $\varphi(x)$ has this property on P .

It should be noted that in the case when the space R is a finite-dimensional Euclidean space, the extension can be carried out in such a way that the exponent α in the Hölder continuity condition does not decrease. For the one-dimensional case this fact is well known, and in the multidimensional case it was recently established in the work of E. M. Landis ⁽²⁾, where the constant C in condition (1) after extension depends essentially on the dimension of the space and, together with it, increases without bound.

Received
8 XII 1964

CITED LITERATURE

1. L. S. Aleksandrov, *Introduction to the General Theory of Sets and Functions*, 1948.
2. E. M. Landis, *Uspekhi Mat. Nauk*, 18, 1 (109), 3 (1963).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.