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Abstract

Full Text

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ON AN ABSTRACT CHARACTERIZATION OF A SIMPLE PROJECTIVE GROUP OF TYPE $PGL(2, K)$ OVER A FIELD K OF CHARACTERISTIC $r \neq 0, 2$

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1. In the last 10 years the following direction has become quite definite in the theory of finite groups: to single out such abstract properties of a group which, under the condition of its simplicity, would characterize certain classes of known simple groups (a class may contain only a finite number of groups). This direction was created in the works of American and Japanese algebraists. First of all, the results (¹⁻¹¹) belong here. In the present article, by means of the concept of a 2-infinitely isolated subgroup, abstract properties are singled out which characterize a simple projective group of type $PGL(2, K)$ over a field K of characteristic $r = 0, 2^*$.
2. **Definition 1.** A subgroup H of a certain group G will be called **2-infinitely isolated** if, from the fact that the centralizer $C_G(h)$ of some element $h \neq 1$ from H in G contains at least one involution from G and has infinite intersection with H , it follows that $C_G(h) \subset H$. Analogously one may introduce the notion of a π -infinitely isolated subgroup, where π is some set of primes.

Definition 2. An element g of a group G is called **approximable** if in the group G there exists a normal divisor of finite index in G not containing the element g , and **nonapproximable** if in the group G there exists no such normal divisor.

Definition 3. An element g of prime order of a group G will be called **almost regular** if $C_G(g)$ is a finite group.

Theorem. *A locally finite group G is then and only then a simple group of type $PGL(2, K)$ over a field K of characteristic $r \neq 0, 2$, when it possesses a 2-infinitely isolated subgroup H satisfying the following conditions: 1) a Sylow 2-subgroup of H is infinite; 2) in the subgroup H there exists an almost regular involution in it, and every involution of this kind from H is nonapproximable in G .*

O. Kegel (¹²) found necessary and sufficient conditions under which a projective group of type $PGL(2, K)$ over a field K of nonzero characteristic is simple (the field K must satisfy these conditions). It is not difficult to verify that in a simple group of type $PGL(2, K)$ the centralizer of any involution in it is a 2-infinitely isolated subgroup and satisfies conditions 1 and 2 of the theorem

formulated above. The sufficiency of the conditions of the theorem follows from the lemmas.

3. Preliminary lemmas.

Lemma 1. *If in a locally finite group some involution is almost regular, then it is approximable in the group.*

Lemma 1 is a new formulation of theorem 5 of the work ⁽¹³⁾.

Lemma 2. *Let a finite group G be a semidirect product of a p -subgroup P and a q -subgroup Q : $G = PQ$ ($p \neq q$), and let the subgroup Q be of type $Q = \{(\{a\} \times \{b\})\lambda\{c\}\}$ with defining relations*

$$a^q = b^q = 1, \quad c^{q^n} = 1, \quad c^{-1}ac = a, \quad c^{-1}bc = ab.$$

* By the field K here is meant the field considered in ⁽¹²⁾.

Then, if the element b induces a regular automorphism in P , the element a is contained in the center of the group G .

In the proof of Lemma 2 the following known result on Sylow primary subgroups of a Frobenius complement is used essentially ⁽¹⁴⁾.

Lemma 3. Let a finite group G be the semidirect product of a p -subgroup P and a q -subgroup Q ($p \neq q$), where the subgroup Q has type

$$Q = \{a\}\lambda\{b\}$$

with defining relations

$$a^{q^n} = b, \quad b^{-1}ab = a^k,$$

where

$$k = k_1q + 1, \quad (k_1, q) = 1.$$

Further, let X be the subgroup of P generated by all elements permutable with the element b , and let P' be a maximal invariant p -subgroup in G , all of whose elements are permutable with a^q . Then, for fixed $p, q, n \geq 1$ and X , the order of the factor group G/P' remains bounded for all finite p -groups (for the given p).

Lemma 4. Let G be an infinite locally finite 2-group, and let \bar{R} be its elementary subgroup of rank ≥ 2 . Then in \bar{R} there exists at least one involution whose centralizer in G is infinite.

Lemma 5. Let G be an infinite locally finite group, and let R be one of its elementary 2-subgroups of rank ≥ 2 . Then in R there exists an involution whose centralizer in G is infinite.

In the proof of Lemma 5, Lemmas 1 and 4 are used essentially.

Lemma 6. Let G be a countable locally finite group, and let p be a prime in $\pi(G)$ ($\pi(G)$ is the set of all prime divisors of the orders of elements of the group G). Then in the group G there exists a Sylow p -subgroup P such that for every finite p -subgroup $P_1 \subset G$ there is an element $g \in G$ such that

$$g^{-1}P_1g \subset P.$$

Lemma 7. If in a locally finite group G , for some $p \in \pi(G)$, all Sylow p -subgroups are extremal, then their ranks are bounded in the aggregate. (In accordance with the survey ⁽¹⁶⁾, a group is called extremal if it is a finite extension of an abelian group satisfying the minimal condition for subgroups.)

In the proof of Lemma 7, Lemma 6 is applied.

Lemma 8. Let G be a locally finite group, and let h be some element of prime order p . Then, if a Sylow p -subgroup in the centralizer $C_G(h)$ is finite, all Sylow p -subgroups of the group G are extremal (for this p)*.

Lemma 9. If some subgroup H of the group G is 2-infinitely isolated, then for every element $g \in G$ the subgroup $g^{-1}Hg$ is also 2-infinitely isolated.

4. Lemmas directly related to the proof of the theorem. In the following lemmas it is assumed throughout that the group G and its subgroup H satisfy the hypotheses of the theorem.

Lemma 10. Let S be a Sylow 2-subgroup of the subgroup H , containing an almost regular involution in H . Then the subgroup S is a finite extension of a quasicyclic group.

In the proof of Lemma 10 the hypotheses of the theorem and Lemmas 1, 5 and 8 are used.

Lemma 11. Let S be an infinite 2-subgroup of the subgroup H , containing an almost regular involution in H . Then the center of the subgroup S is cyclic.

Lemma 12. In the group G there exists a quasicyclic 2-subgroup Q such that

$$H \cap Q = 1.$$

In the proof of Lemma 12, Lemmas 1 and 10, the hypotheses of the theorem, and the definition of a 2-infinitely isolated subgroup are used.

Lemma 13. If the centralizer of an involution $t \in H$ in H is infinite, then the involution t is contained in some quasicyclic subgroup of H .

* Lemmas 6, 7 and 8 were proved jointly with Yu. M. Gorchakov.

Proof. By Lemma 12, in G there exists a quasicyclic subgroup Q such that $H \cap Q = 1$. Let k be an involution from Q . Consider the element $a = tk$. Two

cases are possible: 1) the order of the element a is odd; 2) the order of the element a is even.

Suppose case 1 holds. Then, as is not hard to see, the involutions t and k are conjugate in G , i.e., for some element $g \in G$ the relation $t = g^{-1}kg$ holds, and hence, in view of the choice of the involution k , we have $t \in g^{-1}Qg$. By the condition of the lemma, $C_H(t)$ is infinite, and since H is 2-infinitely isolated, $C_G(t) \subset H$, but then also $g^{-1}Qg \subset H$.

Suppose case 2) holds. In this case, in the subgroup $\langle a \rangle$ there exists an involution j_1 such that $j_1t = tj_1$, $j_1k = kj_1$. Since $C_G(t) \subset H$, we have $j_1 \in H$. If the subgroup $C_H(j_1)$ were infinite, then, by the definition of a 2-infinitely isolated subgroup, we would have $k \in H$, which is impossible. Consequently, $C_H(j_1)$ is a finite group. Let S be a Sylow 2-subgroup containing t and j_1 . By Lemma 10, the subgroup S is a finite extension of a quasicyclic subgroup S' . Let i be an involution from S' . We shall prove that $t = i$. Suppose that $t \neq i$. Consider the centralizer $C_S(S')$. Since the automorphism group of a quasicyclic 2-group is of order 2, the index of $C_S(S')$ in S is equal to 2. Let $t \in C_S(S')$. By assumption, $t \neq i$, and, consequently, the rank of the elementary subgroup $R = \{t, i\}$ is equal to 2, and the centralizer of any involution from R in H is infinite and $R \subset C_G(j_1)$. Since $C_H(j_1)$ is finite, by the hypothesis of the theorem, the involution j_1 is not approximable in G . But then, by Lemma 1, the centralizer $C_G(j_1)$ is infinite. Further, by Lemma 5, the centralizer of some involution $i_1 \in R$ is infinite in $C_G(j_1)$, and since the subgroup H is 2-infinitely isolated and $C_H(i_1)$ is infinite, it follows that $C_G(i_1) \subset H$, and, consequently, the intersection $C_G(j_1) \cap H$ is infinite, which contradicts the finiteness of the subgroup $C_H(j_1)$. It remains to consider the case when $t \notin C_S(S')$. In this case $t = jt_1$, where $t_1 \in C_S(S')$. By hypothesis, $tj_1 = j_1t$. Consequently, t_1 is an involution and $t_1 \in C_G(j_1)$. If $t_1 \neq i$, then, carrying out for the subgroup $R_1 = \{i\} \times \{t_1\}$ arguments analogous to those carried out above for R , we arrive at a contradiction with the finiteness of the centralizer $C_H(j_1)$.

Let $i = t_1$; but then, as is not hard to see, the involutions t and j_1 will be conjugate in $S' \lambda \{j_1\}$, and, consequently, the involution t is almost regular in H , since it is conjugate in H to the almost regular involution j_1 in H . However, this contradicts the condition of the lemma being proved, that $C_H(t)$ is infinite.

All the cases considered have led us to a contradiction with the assumption that $t \neq i$. Consequently, $t = i$, and the lemma is proved.

Lemma 14. Let j be some almost regular involution from H , S a Sylow 2-subgroup of H containing an involution conjugate to j in H , and S' its quasicyclic subgroup. Then the index of the subgroup S' in S remains bounded by some number e for all S of the indicated type.

In the proof of Lemma 14, Lemmas 10 and 11 are used.

Lemma 15. A Sylow 2-subgroup of the group G is either quasicyclic or an infinite dihedral group.

Lemma 15 is proved first for the subgroup H . Let S be a Sylow 2-subgroup of H containing some almost regular involution j . By Lemma 10, the subgroup S is a finite extension of a quasicyclic subgroup S' . If we prove that all involutions from $C_S(S')$ are contained in S' , then it will follow from this that $C_S(S') = S'$ and the subgroup S is a dihedral group. Indeed, if in $C_S(S')$ there existed some element $a \notin S'$, then S' , as a complete abelian subgroup, would split as a direct summand in the abelian subgroup $\{S', a\}$ (17), and then in $C_S(S')$ there would exist an involution outside the subgroup S' , which is impossible. Let t be an involution from $C_S(S')$. By Lemma 13, t is contained in some quasicyclic subgroup $Q \subset H$. Consider the centralizer $C_H(t)$. It contains the subgroups S' and Q . Let $\{c\}$ and $\{d\}$ be cyclic subgroups of sufficiently large orders, taken respectively from the sub-

of the groups S' and Q . For some element $h \in C_H(t)$, in view of the Sylow theorem (17), the subgroup $S_1 = \{h^{-1}ch, d\}$ is a 2-group. We include the subgroup S_1 in a Sylow 2-subgroup \bar{S} of H containing an involution conjugate to j in H . By Lemma 10, the subgroup \bar{S} is a finite extension of a quasicyclic subgroup. In view of Lemma 14, the order of the factor group \bar{S}/\bar{S}' does not exceed some number e , common to all Sylow 2-subgroups of H , each of which contains an involution conjugate to j in H . The elements c and d can be chosen so that their orders are greater than the number e . Consequently $t \in \bar{S}'$, $h^{-1}ih \in \bar{S}'$, where i is an involution from S' . Since \bar{S}' is a quasicyclic group, $h^{-1}ih = t$. Hence $i = h^{-1}th$, and since $h \in C_H(t)$, it follows that $i = t$. In view of the remark made above we obtain that $C_S(S') = S'$, and, consequently, S is a dihedral group. From what has been proved and from the Sylow theorem (17) it is not difficult to obtain that any Sylow 2-subgroup of H is either quasicyclic or a dihedral group. As for the Sylow 2-subgroups of the group G , the validity of the lemma for them follows from its validity for the subgroup H , from the definition of a 2-infinitely isolated subgroup, and from the Sylow theorem.

Lemma 16. *The subgroup H has an invariant 2-complement.*

In the proof of Lemma 16, the hypotheses of the theorem, Lemma 15, the Hall-Grün theorem (17), and also the fact are used that in an infinite dihedral group all involutions outside the quasicyclic subgroup are conjugate to one another.

Lemma 17. *Let h be an element of prime order p from H ($p \neq 2$). Then the centralizer $C_H(h)$ is an infinite group.*

In the proof of Lemma 17, Lemmas 7, 8, 15, 16 are applied; Hall's theorem on automorphisms of a finite p -group P inducing the identity automorphisms in the factor group $P/\Phi(P)$, where $\Phi(P)$ is the Frattini subgroup of the group P (15).

Lemma 18. *In the group G there exists a subgroup H_1 , conjugate to H in G , such that the normalizer of some quasicyclic 2-subgroup \bar{S}' of it in H_1 has the representation*

$$N_{H_1}(\bar{S}') = (\bar{S}' \times A)\lambda\{i\},$$

where A is a subgroup containing no involutions; i is an involution inducing in $\overline{S}' \times A$ an automorphism taking every element from $\overline{S}' \times A$ to its inverse, moreover $i \in H$ and $C_H(i)$ is infinite; for any finite p -subgroup $P_1 \subset H_1$ ($p \neq 2$) there is an element $d \in H_1$ such that $d^{-1}P_1d \subset A$.

In the proof of Lemma 18, Lemmas 2, 3, 7, 8, 9, 13, 15, 16 and 17 are used.

Lemma 19. *Let T be an invariant 2-complement in the subgroup H_1 . Then the subgroup T is abelian. (The subgroup H_1 satisfies the hypotheses of Lemma 18).*

Lemma 19 is proved using Lemmas 13, 15, 17, 18 and the hypotheses of the theorem.

The concluding part of the proof of the theorem rests on the main result of the work ⁽¹¹⁾.

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Note: Figure translations are in progress. See original paper for figures.

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