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Abstract

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On the Investigation of High-Order Nonlinear Automatic Systems by Exact Analytical Methods

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1. The complexity and laboriousness of exact analytical methods for investigating high-order nonlinear systems are well known. However, in a number of cases a nonlinear automatic system of the n-th order can be investigated as completely and simply as is done by means of the phase plane in the case of low-order systems. For this it is necessary to use a method that may naturally be called the method of sections of parameter space, the basic idea of which is as follows. It is established that in the parameter space (of the coefficients) of a nonlinear automatic system of the n-th order, having a linear part and separate nonlinearities, one can construct a series of such sections (usually linear subspaces or hyperplanes) that, for values of the parameters belonging to these sections, the original system is reduced by a nonsingular linear transformation of variables to a series of equations of low order, connected with one another in such a way that they can be integrated successively if some of them are regarded as nonhomogeneous. Thus the difficult problem of investigating the original nonlinear system of the n-th order is reduced, under the conditions of the indicated sections, to the considerably simpler problem of investigating a sequence of low-order systems with the same nonlinearities. Owing to this, the structure of the parameter space (in the sense of the correspondence of points of this space to topologically determined types of the phase portrait of the motion) can be observed in considerable detail in the planes of the sections under consideration. It is assumed that between the indicated sections the system is investigated by other methods.

2. To illustrate what has been said, consider the system

$$\dot{\eta}_k = \sum_{i=1}^{n} a_{ki}\eta_i + \sum_{j=1}^{m} b_k^{(j)} f_j(\sigma_j) \quad (k = 1, \dots, n);$$

$$\sigma_j = \sum_{i=1}^{n} c_i^{(j)}\eta_i \quad (j = 1, \dots, m), \tag{2.1}$$

where $\eta_k,\ \sigma$ are dependent variables; $a_{ki},\ b_k,\ c_i$ are real coefficients; $f_j(\sigma_j)$ are nonlinear functions; the dot denotes differentiation with respect to time t. By means of the transformation

$$\eta_i = -\sum_{k=1}^{n} \frac{H_i(\lambda_k)}{D'(\lambda_k)} x_k \quad (i = 1, \dots, n) \tag{2.2}$$

we reduce system (2.1) to the canonical form

$$\dot{x}_k = \lambda_k x_k + \sum_{j=1}^{m} u_\beta^{(j)}(\lambda_k) f_j(\sigma_j) \quad (k = 1, \dots, n);$$

$$\sigma_j = \sum_{i=1}^{n} \gamma_i^{(j)} x_i \quad (j = 1, \dots, m). \tag{2.3}$$

Here the notation is as follows:

$$H_i(\lambda_k) = \sum_{j=1}^{m} N_i^{(j)}(\lambda_k); \qquad N_i^{(j)}(\lambda_k) = \sum_{\alpha=1}^{m} b_\alpha^{(j)} D_{\alpha i}(\lambda_k); \tag{2.4}$$

$$u_\beta^{(j)}(\lambda_k) = N_\beta^{(j)}(\lambda_k) \left[\sum_{i=1}^{m} N_\beta^{(i)}(\lambda_k) \right]^{-1}; \tag{2.5}$$

$$\gamma_i^{(j)} = -\frac{1}{D'(\lambda_i)} \sum_{k=1}^{n} c_k^{(j)} H_k(\lambda_i) \quad (i = 1, \dots, n;\ j = 1, \dots, m); \tag{2.6}$$

λ_i are the roots of the equation of degree n

$$D(p) \equiv \det \|a_{ki} - \delta_{ki} p\| = 0; \tag{2.7}$$

δ_{ik} is the Kronecker symbol; $D'(\lambda_i) = dD(p)/dp|_{p=\lambda_i}$; $D_{\alpha i}(\lambda_k)$ is the algebraic cofactor of the element in row α, column i, of the determinant $D(p)$ for $p = \lambda_k$. The number β in (2.3) may be different for different k and is chosen so that

$$\sum_{i=1}^{m} N_\beta^{(i)}(\lambda_k) \neq 0.$$

Let $A = \|a_{ki}\|$ be an $n \times n$ matrix, and let ΣH be a column of the form

$$\Sigma H = \left\| \begin{matrix} \sum_{i=1}^{m} b_1^{(i)} \\ \cdots \\ \sum_{i=1}^{m} b_n^{(i)} \end{matrix} \right\|.$$

The nonsingular transformation (2.2) exists if and only if:
1) the roots λ_i $(i = 1, \dots, n)$ are distinct; 2) the vectors $\Sigma H;\ A\Sigma H;\ A^2\Sigma H;\ \dots;\ A^{n-1}\Sigma H$ are linearly independent. In what follows we assume both these conditions to be satisfied.

3. Consider the case when, in (2.1), the quantities $c_i^{(j)}$ $(i = 1, \dots, n;\ j = 1, \dots, m)$ are parameters, while the coefficients $a_{ki}, b_k^{(j)}$ are prescribed numbers. We shall interpret the space $C_n^{(j)}$ as an n-dimensional Euclidean space generated by n-dimensional vectors whose components in some orthonormal basis are the real numbers $c_k^{(j)}$ $(k = 1, \dots, n)$. The space of $n \times m$ parameters $c_i^{(j)}$ is composed of m spaces $C_n^{(j)}$ $(j = 1, \dots, m)$. Let us write m systems of linear equations of the form

$$\sum_{k=1}^{n} c_k^{(1)} H_k(\lambda_i) = A_i^{(1)} \quad (i = 1, \dots, n);$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{3.1}$$

$$\sum_{k=1}^{n} c_k^{(m)} H_k(\lambda_i) = A_i^{(m)} \quad (i = 1, \dots, n),$$

each of which has n unknowns $c_k^{(j)}$ $(k = 1, \dots, n)$. The matrix $\|H_k(\lambda_i)\|$, formed from the coefficients of the unknowns of any of the systems (3.1), has a determinant proportional to the determinant of the transformation matrix (2.2). Consequently, in the case under consideration, when the transformation (2.2) is nonsingular, the systems (3.1) are consistent independently of the choice of the numbers $A_i^{(j)}$.

Let in (3.1)

$$A_i^{(j)} = 0 \quad \text{for } i = 1, \dots, s-1,\ s+1, \dots, n \quad (j = 1, \dots, m), \tag{3.2}$$

and let $A_s^{(j)}$ $(j = 1, \dots, m)$ be arbitrary real constants. Then the systems (3.1) determine the values of the parameters $C_n^{(j)}$ belonging to of the set of spaces $C_n^{(j)}$ by a section of the first kind $G_1^{(s)}$, for which, according to (2.6), in equations (2.3) $\gamma_i^{(j)} = 0$ for $i \neq s$ and only $\gamma_s^{(j)} \neq 0$ $(j = 1, \dots, m)$. Therefore, under the conditions of the section $G_1^{(s)}$, the variables σ_j $(j = 1, \dots, m)$ will be determined from an independent first-order system of the form

$$\dot{x}_s = \lambda_s x_s + \sum_{j=1}^{m} u_\beta^{(j)}(\lambda_k) f_j(\sigma_j), \qquad \sigma_j = \gamma_s^{(j)} x_s \quad (j = 1, \dots, m). \tag{3.3}$$

Having determined from (3.3) $\sigma_j(t)$ and $x_s(t)$, the remaining $n-1$ equations (2.3) for x_k, $k \neq s$, will already be regarded as linear nonhomogeneous first-order equations. Taking λ_s to be any of the s real roots of equation (2.7), we compose s conditions of the form (3.2) and form s different sections $G_1^{(s)}$. Each section $G_1^{(s)}$ will be an m-dimensional real space, represented in each of the spaces $C_n^{(j)}$ $(j = 1, \dots, m)$ by a subspace of dimension 1, i.e., by a straight line.

Let now, in (3.1),

$$A_i = 0 \quad \text{for } i = 1, \dots, s-1,\ s+1, \dots, r-1,\ r+1, \dots, n, \tag{3.4}$$

while A_s and A_r are real (if λ_s and λ_r are real) or complex conjugates (if λ_s and λ_r are complex conjugates), and otherwise arbitrary constants. Under these conditions, the systems (3.1) determine values $c_k^{(j)}$ belonging, in the set of spaces $C_n^{(j)}$, to a section of the second kind $G_2^{(s,r)}$, for which in (2.3) $\gamma_i^{(j)} = 0$ for $i \neq s$, $i \neq r$, and only $\gamma_s^{(j)} \neq 0$, $\gamma_r^{(j)} \neq 0$ $(j = 1, \dots, m)$. Therefore, under the conditions of the section $G_2^{(s,r)}$, the variables σ_j are determined from an autonomous second-order system

$$\dot{x}_l = \lambda_l x_l + \sum_{j=1}^{m} u_\beta^{(j)}(\lambda_l) f_j(\sigma_j) \quad (l = s, r),$$

$$\sigma_j = \gamma_s^{(j)} x_s + \gamma_r^{(j)} x_r \quad (j = 1, \dots, m), \tag{3.5}$$

and, after determining $\sigma_j(t)$, $x_s(t)$, $x_r(t)$, the remaining $n-2$ equations (2.3) for the variables x_k, $k \neq s$, $k \neq r$, can again be regarded as linear nonhomogeneous first-order equations. Taking λ_s, λ_r to be any two of the s real roots or any pair among the $q/2$ pairs of complex conjugate roots of equation (2.7), in all we construct $0.5[s(s-1)+q]$ different sections $G_2^{(s,r)}$, each of which will be a $2m$-dimensional real space, represented in each of the spaces $C_n^{(j)}$ $(j = 1, \dots, m)$ by a subspace of dimension 2, i.e., by a plane.

Let us emphasize that the dimension of the phase space and the order of the system under study under the conditions of the sections considered remain equal to n.

The bifurcation surfaces obtained in the sections $G_1^{(s)}$, $G_2^{(s,r)}$ will be the intersections of the bifurcation surfaces of the space of $n \times m$ parameters $c_i^{(j)}$ with the named sections.

Everything set forth remains valid when, in (2.1), $m = 1$.

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Note: Figure translations are in progress. See original paper for figures.