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Abstract

Full Text

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TESTS, UNBIASED ESTIMATES, AND COTEST IDEALS

Let some space of elementary events $\mathfrak{X}$ be endowed with a σ-algebra $\mathfrak{B}$ and with a family of probability measures $P = \{p_\theta,\ \theta \in \Omega\}$, where θ belongs to some subset of the real Euclidean space E_s, and the measures p_θ are dominated by one measure. We shall consider problems of testing the null hypothesis $H_0:\ \theta \in \omega$ (ω a proper nonempty subset of Ω) against the alternative $H_1:\ \theta \in \Omega \setminus \omega$, and also the construction of unbiased estimates χ for some projection θ_i of the vector parameter $\theta = (\theta_1, \dots, \theta_s)$.

Tests for H_0 will be regarded as randomized and similar of level α: a test $\Phi = \Phi(x)$ will have the property

$$E(\Phi \mid \theta_0) = \alpha \quad \text{for } \theta_0 \in \omega,\ x \in \mathfrak{X}. \tag{1}$$

A **cotest** under these conditions will mean any function ψ of the form $\psi = A(\Phi - \alpha)$, where A is a constant; obviously, then $E(\psi \mid \theta_0) = 0$ for $\theta_0 \in \omega$; a **precotest** is any statistic $\xi(x)$, measurable with respect to all p_θ, under the condition $E(\xi(x) \mid \theta_0) = 0$ for $\theta_0 \in \omega$. An unbiased estimate χ of some component of θ, say θ_1, behaves, obviously, analogously to a precotest: $E(\chi - \theta_1) = 0$ for all $\theta \in \Omega$, so that one may say that a precotest is an "unbiased estimate of zero" for $\theta_0 \in \omega$.

Let the family of measures $\{p_\theta,\ \theta \in \omega\}$ be complete in the sense, slightly strengthening the definition of Lehmann–Scheffé ([1,2]): if for some measurable function $f(x)$ there exist and are given, for all θ, its "images" :

$$E(f(x) \mid \theta) = \varphi(\theta), \tag{2}$$

then from them one can reconstruct the "preimage" $f(x)$ with probability 1 (in particular, in this case the unbiased estimate χ is unique).

Among measurable functions $f(x)$ we shall consider, when they exist, such classes of functions $\mathfrak{K}$ for which the following requirements are fulfilled:

For $f \in \mathfrak{K}$ the image $\varphi(\theta) = E(f(x) \mid \theta)$ is a holomorphic function in some polycylinder Z of the complex Euclidean space E_{2s}, and the images of the functions $f \in \mathfrak{K}$ form a ring K_θ with operators from the field of complex numbers. The totality of functions $\varphi(\theta) \in K_\theta$ that vanish for $\theta \in \omega$ forms an ideal I_{H_0}. All of them, obviously, are precotests, and we shall for brevity call I_{H_0} a **cotest ideal** (although the more exact name would be **precotest ideal**). If the ring K_θ is sufficiently broad (which will be the case, in particular, for exponential families of distributions), then one may try to describe the cotest ideal I_{H_0} by finding its basis, and then, with the aid of this description, seek a cotest ψ corresponding to an optimal, in a certain sense, test Φ of level α. We may, for example, introduce a Bayes measure $B(\theta)$ on the set $\Omega \setminus \omega$ corresponding to the alternative H_1, and seek a precotest under the conditions $-\alpha \le \xi \le 1-\alpha$, which would give the greatest possible value of the functional

$$\int_{(\Omega\setminus\omega)} E(\xi \mid \theta)\, dB(\theta).$$

The indicated conditions make the pre-cotest ξ a cotest. A cotest ψ_ε for which

$$\int_{(\Omega\setminus\omega)} E(\psi_\varepsilon \mid \theta)\, dB(\theta) \ge \sup_\psi \int_{(\Omega\setminus\omega)} E(\psi \mid \theta)\, dB(\theta) - \varepsilon, \tag{3}$$

where the supremum is taken over all cotests of level α, will be called ε-optimal.

The description of ideals in rings of holomorphic functions requires the apparatus of the theory of coherent analytic sheaves, developed in ([3]). At present sufficiently complete results are available only for ideals of the ring of all functions holomorphic in a certain simply connected compact polycylinder of the space $\bar{E}_{2s}$ ([4,5]). In the aspect considered by us, rings of this type are applicable, for example, in considering exponential families of distributions in a natural parametrization (([6]), p.51). Here we have:

$$p_\theta = C(\theta) \exp\left(\sum_{j=1}^{s} \theta_{jT} j\right) h(T_1, \dots, T_s), \tag{4}$$

where $T_1, \dots, T_s$ are scalar sufficient statistics for θ; $h(T_1, \dots, T_s)$ is a density with respect to Lebesgue measure.

If, for simplicity, we assume that the sufficient statistics $T_1, \dots, T_s$ are nonnegative and $h(T_1, \dots, T_s) > 0$, then one may consider the ring of all functions holomorphic in the product $R_1 \times \dots \times R_s$ of half-planes $R_i : \operatorname{Re}\theta_i > 0$.

Let now the set $\omega \subset \Omega$, corresponding to the hypothesis H_0, be described by $r < s$ conditions:

$$\Pi_1\left(\frac{1}{\theta_1}, \dots, \frac{1}{\theta_s}\right) = 0; \dots; \quad \Pi_2\left(\frac{1}{\theta_1}, \dots, \frac{1}{\theta_s}\right) = 0, \tag{5}$$

where the functions Π_j are holomorphic for $\theta_i \neq 0$; the θ_i are real and subject to the condition $\theta_1 > \varepsilon_0, \theta_s > \varepsilon_0$, where ε_0 is a small positive number. Then, under certain additional conditions imposed on the relations (5), it is possible to describe a basis of the pre-cotest ideal I_H, and to reduce the problem of the optimal test Φ of level α (for a given Bayesian measure $B(\theta)$) to problems of linear programming. The situation described occurs, for example, in the construction of tests of the "even type" for a linear hypothesis in the method of least squares with unknown weights—in particular, the test for the Behrens-Fisher problem with sufficient $\bar{x}, \bar{y}, s_1^2, s_2^2$ (in the usual notation), depending only on $|\bar{x} - \bar{y}|, s_1^2, s_2^2$. In order to formulate one of the theorems relating to this question, replace in condition (5) θ_l by p_l^{-1}, so that we obtain the conditions $\Pi_j(p_1, \dots, p_l) = 0$ $(j = 1, 2, \dots, r)$; here $p_l = x_l + iy_l$ will lie in the disk $(x_l - R^2) + y_l^2 \leq R^2$ for some sufficiently large R. The product of these disks forms a polycylinder Z. We shall further require that the analytic set $\{\Pi_j = 0,\ j = 1, 2, \dots, r\}$ split in Z into a finite number of connected analytic sets, each of real dimension $2(s-r)$ and each containing within itself a real analytic set of real dimension $s - r$.

Next, we require that the set of points of Z under the conditions

$$\Pi_1(p_1, \dots, p_s) = 0, \dots, \Pi_r(p_1, \dots, p_s) = 0; \qquad \operatorname{rang} \left\| \begin{matrix} \dfrac{\partial \Pi_1}{\partial p_1} \cdots \dfrac{\partial \Pi_1}{\partial p_s} \\ \cdots\cdots\cdots \\ \dfrac{\partial \Pi_r}{\partial p_1} \cdots \dfrac{\partial \Pi_s}{\partial p_s} \end{matrix} \right\| < r$$

consisted of a finite number of points. Under these conditions, from K. Oka' s theorem ((5), p. 26; (4)) one can deduce that the cotest ideal I_{H_0}, defined by the relations (5) in the real domain, will have a basis $\Pi_1, \Pi_2, \dots, \Pi_r$; this basis will have easily computable prototypes—pretests $F_1, F_2, \dots, F_r$. We can now formulate the theorem.

Theorem. Let the conditions (5), describing the null hypothesis H_0, be subject to the requirements explained above. Let the density $h(T_1, \dots, T_s)$ in formula (4) be differentiable and satisfy the conditions:

$$h(T_1, \dots, T_s) = O(\exp \varepsilon (T_1 + \dots + T_s)) \qquad (\varepsilon > 0 \text{ arbitrarily small}),$$

$$T_j > 0 \ (j = 1, 2, \dots, s), \qquad \partial \ln h / \partial T_j = O\left(\frac{1}{T_1} + \dots + \frac{1}{T_s}\right),$$

if some $T_j \to \infty$. Then the ε-optimal cotests can be given in the form

$$\psi = \frac{1}{h}\{F_1 * G_1 + \dots + F_r * G_r\}, \tag{6}$$

where $G_j = G_j(x)$ are smooth functions, and $*$ is the convolution operation for the one-sided Laplace transform.

For example, in order to find an ε-optimal similar test in the Behrens-Fisher problem, depending only on $|\bar{x} - \bar{y}|, s_1^2, s_2^2$, one must find a smooth function $H(T_1, T_2, \dots, T_s)$ such that

$$-\alpha \leq (F_1 * H) T_1^{1/2} T_2^{-m_1} T_3^{-m_2} \leq 1 - \alpha,$$

and for which the cotest

$$\psi = (F_1 * H) T_1^{1/2} T_2^{-m_1} T_3^{-m_3}$$

makes the functional (3) as large as possible. Here $F_1 = n_2 T_1 + n_1 T_2 - T_3$; n_1, n_2 are the sample sizes; $m_i = \frac{1}{2}(n_i - 3)$ $(i = 1, 2)$. Here the ideal I_{H_0} is principal (generated by one element)*. The indicated restrictions can be substantially weakened and a description of ε-optimal similar tests can be given for significant classes of families. By the same method one can study questions of optimality and "ε-optimality" of unbiased estimators.

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* In the terminology of algebraic number theory.

Note: Figure translations are in progress. See original paper for figures.