FREQUENCY CONDITIONS FOR ABSOLUTE STABILITY OF PULSE AUTOMATIC SYSTEMS WITH ONE NONLINEAR BLOCK

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Abstract

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MATHEMATICS

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Pulse control systems with a finite number of degrees of freedom and one nonlinear block can be described by difference equations of the form

$$x_{t+1} = Px_t + q\varphi(\sigma_t), \qquad \sigma_t = r^* x_t, \qquad t = 0, 1, \dots, \tag{1}$$

where x, q, r are real $v \times 1$ vectors (the asterisk denotes transposition); $\varphi(\sigma)$ is a scalar, continuous, real function of σ; P is a real $v \times v$ matrix.

Assume that:

A. All eigenvalues π_j of the matrix P satisfy the conditions

$$|\pi_j| \leq 1, \quad j = 1, \dots, v.$$

B. The vectors $q, Pq, \dots, P^{v-1}q$ are linearly independent.

C. $\varphi(0) = 0$, and for any $\sigma_1 \neq \sigma_2$ the inequalities

$$\alpha_1 \leq (\sigma_2 - \sigma_1)^{-1} [\varphi(\sigma_2) - \varphi(\sigma_1)] \leq \alpha_2$$

hold, where $\alpha_1 \leq 0,\ \alpha_2 > 0$.

D. If $|\pi_j| < 1,\ j = 1, \dots, v$ (the noncritical case), then

$$0 \leq \sigma^{-1}\varphi(\sigma) \leq \mu_0$$

for all $\sigma \neq 0$, where $0 < \mu_0 \leq \alpha_2$; if for at least one value of j one has $|\pi_j| = 1$ (the critical case), then

$$0 < \varepsilon \leq \sigma^{-1}\varphi(\sigma) \leq \mu_0$$

for all $\sigma \neq 0$, where $\varepsilon < \mu_0 \leq \alpha_2$.

By the amplitude-phase characteristic $\chi(\lambda) = r^*(P - \lambda I)^{-1}q$ (where $|\lambda| = 1$) of the linear part of system (1), define two functions

$$\Phi_k(\lambda,\vartheta,\xi) = \mu_0^{-1} + \operatorname{Re}\chi(\lambda) - \vartheta(-1)^k\left[\operatorname{Re}(1-\lambda)\chi(\lambda) + \frac{1}{2}\alpha_k|(1-\lambda)\chi(\lambda)|^2\right] + \xi|1-\lambda|^2\left[1 + (\alpha_1+\alpha_2)\operatorname{Re}\chi(\lambda) + \alpha_1\alpha_2|\chi(\lambda)|^2\right], \qquad k = 1,2; \tag{2}$$

here $\alpha_1, \alpha_2, \mu_0$ are the same as in conditions C, D.

Theorem 1. Let $|\pi_j| < 1,\ j = 1, \dots, v$. Suppose that, for some $\vartheta \geq 0,\ \xi \geq 0$ and all $\lambda,\ |\lambda| = 1$, either

$$\Phi_1(\lambda,\vartheta,\xi) > 0,$$

or

$$\Phi_2(\lambda,\vartheta,\xi) > 0$$

is satisfied.

Then the trivial solution $x_t \equiv 0$ of system (1) is asymptotically stable in the large.

Theorem 2. Let $|\pi_j| \leq 1,\ j = 1, \dots, v$, and suppose there exists μ_*, $\varepsilon \leq \mu_* \leq \mu_0$, for which the spectrum of the matrix

$$P_* = P + \mu_* q r^*$$

is entirely located inside the unit circle*.

Suppose that, for some $\vartheta \geq 0,\ \xi \geq 0$ and all $\lambda,\ |\lambda| = 1$, either

$$\Phi_1(\lambda,\vartheta,\xi) \geq 0,$$

or

$$\Phi_2(\lambda,\vartheta,\xi) \geq 0$$

is satisfied. Then the trivial solution $x_t \equiv 0$ of system (1) is asymptotically stable in the large.

The question of the existence of the numbers ϑ, ξ indicated in Theorems 1 and 2 can be resolved graphically, analogously to ([1]).

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* The existence of the indicated number μ_* is necessary for the assertion of Theorem 2 to be valid.

Consider special cases. 1) If, in conditions C, $\alpha_1 = -\infty$, $\alpha_2 = +\infty$, then necessarily $\vartheta = \xi = 0$, and the inequality $\Phi_k(\lambda, 0, 0) \equiv \mu_0^{-1} + \operatorname{Re}\chi(\lambda) > 0$ is the well-known criterion of Ya. Z. Tsypkin ([2]).* 2) If, in relations C, $-\alpha_1 = \alpha_2 < +\infty$, then the frequency condition $\Phi_k(\lambda,\vartheta,0) \geqslant 0$ ($k = 1$ or $k = 2$) coincides with the criterion of Segré ([4]), obtained for the noncritical case under the assumptions $0 \leqslant \sigma^{-1}\varphi(\sigma) \leqslant \mu_0$, $0 \leqslant d\varphi/d\sigma < \alpha_2$. 3) For the case $\mu_0 = \alpha_2 < +\infty$, the stability condition $\Phi_2(\lambda, \vartheta, 0) \geqslant 0$ was obtained by Segré in ([5]). 4) If, in relations C, $\alpha_2 = +\infty$, $\alpha_1 > -\infty$, then Segré' s criteria ([4,5]) necessarily coincide with the Segré–Kalman condition ([3]) and are less broad than the frequency conditions of Theorems 1 and 2 of the present paper. 5) In the case $\alpha_1 = 0$, Theorems 1 and 2 include, as is easily verified, the corresponding results ([6,7]) (Theorem 5 ([7])) of the author and, consequently (see ([7])), the results ([3–5]).

It can be shown that, in cases 2), 3), the conditions $\Phi_k(\lambda, \vartheta, \xi) \geqslant 0$ ($\Phi_k > 0$), $\vartheta > 0$, $\xi > 0$, may be satisfied when the conditions $\Phi_k(\lambda, \vartheta, 0) \geqslant 0$ are certainly not valid for any $\vartheta \geqslant 0$.

The proofs of Theorems 1 and 2 are carried out by the method of matrix inequalities of V. A. Yakubovich (see, for example, ([1,8,9])), modified for the difference-equation case under consideration, and are based on the following algebraic proposition:

Theorem 3. Let e, s, and $Q, G_0 = G_0^*$ be, respectively, some given real $\nu \times 1$ vectors and $\nu \times \nu$ matrices, and let ξ be a real number. Suppose that: 1) one eigenvalue of the matrix Q is equal to unity, while the others are located inside the unit circle; 2) the vectors $e, Qe, \dots, Q^{\nu-1}e$ are linearly independent. Define, in terms of the unknown real $\nu \times \nu$ matrix $H = H^*$, the quadratic form $\Psi(z, \zeta)$, where z and ζ are a real $\nu \times 1$ vector and a number, by the relations

$$G = H - Q^*HQ, \qquad -g = Q^*He + s, \qquad \gamma = \xi - e^*He,$$

$$\Psi(z, \zeta) = z^*(G + G_0)z + 2g^*z\zeta + \gamma\zeta^2 \tag{3}$$

and denote

$$e_\lambda = (Q - \lambda I)^{-1}e, \qquad \Phi_0(\lambda) = \xi + 2\operatorname{Re} s^*e_\lambda + e_\lambda G_0 e_\lambda. \tag{4}$$

For the existence of a matrix $H = H^*$ satisfying the inequality $\Psi(z, \zeta) > 0$ for all z, ζ, $|z| + |\zeta| \neq 0$, it is necessary and sufficient that the following hold: a) $\Phi_0(\lambda) > 0$ for all $\lambda \neq 1$, $|\lambda| = 1$; b) $\lim_{\lambda \to 1} |1 - \lambda|^2 \Phi_0(\lambda) > 0$.

The **proof of Theorem 3** can be obtained with the aid of Lemma 1 ([8]), if one sets $Q = (I + A)(I - A)^{-1}$, $\lambda = (1 + i\omega)(1 - i\omega)^{-1}$, $\omega \geqslant 0$, and carries out the nonsingular transformation $z = \sqrt{2}(Q + I)^{-1}(z_1 - \zeta_1 e)$, $\zeta = i\sqrt{2}\zeta_1$. The problem under consideration is thereby reduced to the solution of a matrix inequality (Theorem 4, ([1])).

Proof of Theorem 1. Denote

$$\varphi_t = \varphi(\sigma_t), \quad \Delta\sigma_t = \sigma_{t+1} - \sigma_t, \quad \zeta_t = \zeta(z_t) = \Delta\varphi_t = \varphi_{t+1} - \varphi_t, \tag{5}$$

$$z_t = \left\| \begin{matrix} x_t \\ \varphi_t \end{matrix} \right\|, \quad Q = \left\| \begin{matrix} P & q \\ 0 & 1 \end{matrix} \right\|, \quad e = \left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\|, \quad r_1 = \left\| \begin{matrix} r \\ 0 \end{matrix} \right\|, \quad r_2 = \left\| \begin{matrix} (P^* - I)r \\ r^* q \end{matrix} \right\|.$$

In view of (5), system (1) is equivalent to the equations

$$z_{t+1} = Qz_t + e\zeta(z_t), \qquad t = 0, 1, \ldots \tag{6}$$

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* For systems of the form (1), under the assumptions $|\pi_j| < 1,\ j = 1, \ldots, \nu,\ 0 \leqslant \sigma^{-1}\varphi(\sigma) < \mu_0$, an analogous condition with the strict inequality replaced by $\mu_0^{-1} + \operatorname{Re}\chi(\lambda) \geqslant 0$ was obtained by Kalman and Segré ([3]). (Here and below, the notation introduced in the cited papers has been replaced by the notation of the present paper.)

Choose a real $(\nu + 1) \times (\nu + 1)$ matrix $H = H^*$ and a number $\vartheta \geqslant 0$ such that the function

$$V_k(z) = z^* H z + \vartheta(-1)^k \int_0^\sigma \varphi(\sigma')\, d\sigma' \quad (\sigma = r_1^* z), \tag{7}$$

where $k = 1$ or 2, satisfies the conditions of Lemma 1 ([6]). Condition (I) of Lemma 1 ([6]) is, obviously, satisfied.

From (7), by virtue of (5), (6), we have

$$\Delta V_k(z_t) \equiv V(z_{t+1}) - V(z_t) =$$

$$= -\Psi(z_t, \xi_t) - \Omega_1(\sigma_t^{-1}\varphi_t)\sigma_t^2 - \vartheta\Omega_1^{(k)}(\sigma_t, \Delta\sigma_t) - \xi\Omega_3(\Delta\sigma_t^{-1}\Delta\varphi_t)\Delta\sigma_t^2.$$

Here

$$\Omega_1(\mu) = \mu(1 - \mu_0^{-1}\mu); \quad \Omega_1^{(k)}(\sigma, \Delta\sigma) =$$

$$= (-1)^k \left[\varphi(\sigma)\Delta\sigma + \frac{1}{2}\alpha_k \Delta\sigma^2 - \int_\sigma^{\sigma+\Delta\sigma} \varphi(\sigma')\, d\sigma' \right]; \quad \Omega_3(\alpha) = (\alpha - \alpha_1)(\alpha_2 - \alpha);$$

the numbers $\mu_0, \alpha_1, \alpha_2$ are the same as in conditions C, D; the real parameter $\xi \geqslant 0$ is to be chosen, and the quadratic form $\Psi(z, \zeta)$ is determined by relations (3), where

$$s = \tfrac{1}{2}\xi(\alpha_1+\alpha_2)r_2, \quad G_0 \equiv G_0^{(k)} = \mu_0^{-1}ee^* - \tfrac{1}{2}(er_1^* + r_1e^*) -$$

$$-\tfrac{1}{2}\vartheta(-1)^k(er_2^* + r_2e^*) + \left[\xi\alpha_1\alpha_2 - \tfrac{1}{2}\vartheta(-1)^k\alpha_k\right] r_1r_2^*, \quad k=1,2. \qquad (8)$$

For convenience of the estimates, in the expression for $\Delta V_k(z_t)$, obtained from (6), (7), the quantity

$$[\varphi_t(\sigma_t - \mu_0^{-1}\varphi_t) + \vartheta(-1)^k(\varphi_t\Delta\sigma_t + \tfrac{1}{2}\alpha_k\Delta\sigma_t^2) +$$

$$+\xi(\Delta\varphi_t - \alpha_1\Delta\sigma_t)(\alpha_2\Delta\sigma_t - \Delta\varphi_t)],$$

transformed taking into account relations (5), has been added and subtracted.

Suppose that there exist numbers $\vartheta \geqslant 0,\ \xi \geqslant 0$, satisfying the frequency condition $\Phi_k > 0$ of Theorem 1 ($k=1$, or $k=2$). According to requirements C, D we have $\Omega_j \geqslant 0,\ j=1,2,3$. From relations (2), (4), by virtue of (5), (8), we obtain $|1-\lambda|^2\Phi_0(\lambda) \equiv \Phi_k(\lambda,\vartheta,\xi)$. It is easy to verify that the matrix Q and the vector e satisfy requirements 1), 2) of Theorem 3, and, by the assertion of this theorem, from the condition $\Phi_k > 0$ there follows the existence of a matrix $H = H^*$ for which $\Psi(z,\zeta) > 0$ for any $z, \zeta,\ |z| + |\zeta| \neq 0$. Taking (5) and the estimates for Ω_j into account, we obtain $\Delta V_k(z) < 0$ for all $z \neq 0,\ \Delta V_k(0) = 0$. Thus, the function (7) satisfies conditions (III), (IV) of Lemma 1 ([6]).

Denote $Q_\mu = Q + \mu e r_2^*,\ H_k(\mu) = H + \frac{1}{2}\vartheta(-1)^k\mu r_1 r_2^*,\ G_k(\mu) = H_k(\mu) - Q_\mu^* H_k(\mu) Q_\mu,\ k=1,2$, and consider the particular case $\varphi(\sigma) = \mu\sigma,\ \mu \in [0,\mu_0]$^*. From (5), (6), (7) we have $z_{t+1} = Q_\mu z_t,\ V_k(z) = z^* H_k(\mu) z,\ \Delta V_k(z) = -z^* G_k(\mu) z$. By what has been proved, $G_k(\mu) > 0$ for arbitrary $\mu,\ 0 \leqslant \mu \leqslant \mu_0$, and, by continuity, $G_k^{(\varepsilon)}(\mu) \equiv H_k(\mu) - Q_\mu(\varepsilon)^* H_k(\mu) Q_\mu(\varepsilon) > 0$ for all $\mu \in [0,\mu_0]$, where $Q_\mu(\varepsilon) = Q_\mu - \varepsilon ee^*$, and ε is a sufficiently small positive number. From the frequency condition $\Phi_k(\lambda,\vartheta,\xi) > 0,\ \vartheta \geqslant 0,\ \xi \geqslant 0$ of Theorem 1 it follows that the hodograph $\chi(\lambda)$ does not intersect the real axis on the interval $(-\infty, -\mu_0^{-1}]$ when $|\lambda| = 1$. Therefore the spectra of all matrices $P_\mu = P + \mu q r^*,\ \mu \in [0,\mu_0]$, lie inside the unit circle (see also the conditions of Theorem 1 on π_j). With the aid of the rela–

* Obviously, the function $\varphi(\sigma) = \mu\sigma$ satisfies conditions C, D for any $\mu,\ 0 \leqslant \mu \leqslant \mu_0$. Therefore, for the case under consideration, all the estimates obtained above are valid.

relations (5) it is easy to show that the spectra of all matrices $Q_\mu(\varepsilon),\ \mu \in [0,\mu_0]$, for sufficiently small $\varepsilon > 0$ are also located inside the unit circle. From the inequality $G_k^{(\varepsilon)}(\mu) > 0$, by Lemma 3 ([6]), it follows that $H_k(\mu) > 0$ for all $\mu,\ 0 \le \mu \le \mu_0$. Represent the function (7) in the form

$$V_k(z) = z^* H_k(\mu_k) z + \vartheta \int_0^{r_1^* z} [\mu_k + (-1)^k \sigma^{-1} \varphi(\sigma)] \sigma \, d\sigma,$$

where $\mu_1 = \mu_0,\ \mu_2 = 0$. By condition D the second term in the expression $V_k(z)$ is nonnegative, and, since $H_k(\mu) > 0,\ \mu \in [0, \mu_0]$, we obtain $V_k(z) > 0$ for $z \neq 0$, $\lim_{|z|\to\infty} V(z) = +\infty$, i.e., the function (7) satisfies condition II of Lemma 1 ([6]), from which the assertion of Theorem 1 follows.

The **proof of Theorem 2** can be obtained by reducing the problem to the noncritical case (Theorem 1) by means of the substitution $\varphi(\sigma) = \mu_0 \sigma - \varphi_1(\sigma)$, as was done, for example, in ([10]).

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Note: Figure translations are in progress. See original paper for figures.