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Academician of the Academy of Sciences of the MSSR V. A. ANDRUNAKIEVICH, Yu. M. RYABUKHIN

PRIMARY IDEALS IN NONCOMMUTATIVE RINGS

The concept of a primary ideal originally arose in Noetherian rings, i.e., in commutative rings satisfying the maximal condition for ideals. In works ([1–6]) this concept was carried over to noncommutative rings for the purpose of constructing an additive theory of ideals. In the present note a general definition of primarity is given and its connection with the older, more special notions of primarity is established. It is proved that, under certain natural requirements imposed on primarity, the notion of a tertiary ideal introduced by Lesieur and Croisot ([6]) is the only possible generalization of the classical notion of a primary ideal.

In what follows, by a ring we shall mean an associative, not necessarily commutative ring satisfying the maximal condition for (two-sided) ideals. If A and B are ideals of a ring K, then the ideal $B^r_A = \{x \mid x \in K,\ Bx \subseteq A\}$ will be called the **right annihilator of the ideal** B **modulo** A. In general, by a **right annihilator modulo** A we shall mean an ideal C such that $C = B^r_A$ for a suitable ideal B. If $B \nsubseteq A$, then B^r_A will be called a right proper annihilator modulo A. The left annihilator B^l_A of the ideal B modulo A is defined analogously. We shall say that an s-primarity S is defined in a ring K if with each ideal $Q \neq K$ there is associated an ideal $s(Q) \supseteq Q$ such that $s^{(r)}_Q(Q) \supset Q$, where $s^r_Q(Q) = [s(Q)]^r_Q$. It follows from this that if $B \subseteq s(Q)$, then $B^r_Q \supset Q$. We shall call an ideal Q s-**primary** if from $B^r_Q \supset Q$ it follows that $B \subseteq s(Q)$, or, what is the same, from $BC \subseteq Q$, $C \nsubseteq Q$ it follows that $B \subseteq s(Q)$. We assume that K itself is an s-primary ideal in K and that $s(K) = K$. We shall say that an s_1-primarity S_1 is stronger than an s_2-primarity S_2 if every s_1-primary ideal is also an s_2-primary ideal. Finally, we shall say that an s_1-primarity S_1 is equivalent to an s_2-primarity S_2 if the s_1-primary ideals are precisely the s_2-primary ideals. The notion of s-primarity S is a generalization of certain known types of primarity: primaryness Π ([2,6]), primarity P_r ([3,6]), tertiarity T ([6]), simplicity P, irreducibility N, and others. Recall that an ideal Q is called **primary** if from $A^r_Q \supset Q$, $B^r_Q \supset Q$ it follows that $(A + B)^r_Q = Q$. It is known ([6]) that an ideal Q is primary if and only if from $B^r_Q \supset Q$ it follows that $B \subseteq \pi(Q)$, where $\pi(Q) = \bigcap\{R^l_Q \mid R^l_Q$ are maximal proper left annihilators modulo $Q\}$. An ideal Q is simple if and only if from $B^r_Q \supset Q$ it follows that $B \subseteq Q$.

Proposition 1. *Let S be an arbitrary s-primarity. Every s-primary ideal is primary. Every simple ideal is s-primary. Primaryness Π is the weakest s-primarity, and simplicity P is the strongest s-primarity.*

For the proof, put, for any ideal $Q \neq K$,

$$\pi(Q) = \bigcap\{R^l_Q \mid R^l_Q \text{ are maximal proper left annihilators modulo } Q\}$$

and $p(Q) = Q$. This defines the π-primarity Π and the p-primarity P, which coincide respectively with primaryness and simplicity. Next it is verified that the properties Π and P satisfy the conditions of Proposition 1.

Proposition 2. Let S be an arbitrary s-primarity. If Q is an s-primary ideal, then $s(Q)$ will be a prime ideal.

Indeed, let $A \supset s(Q)$, $B \supset s(Q)$. Then $A^r_Q = Q$, $B^r_Q = Q$, whence

$$(AB)^r_Q = B_{A^r_Q} = B^r_Q = Q.$$

Consequently, $AB \not\subset s(Q)$.

In what follows we shall say that an s-primary ideal belongs to the prime ideal $s(Q)$.

Proposition 3. Let S be an arbitrary s-primarity. Then for every s-primary ideal Q the equality $s(Q) = s^{rl}_Q(Q)$ holds. An ideal $Q \neq K$ is s-primary if and only if $s(Q)$ is the unique maximal left proper annihilator modulo Q.

Proof. Let $Q \neq K$ be an s-primary ideal. Since always $s^{rlr}_Q(Q) = s^r_Q(Q)$ and $s^r_A(Q) \supset Q$, by the s-primarity of Q, $s^{rl}_Q(Q) \subseteq s(Q)$. On the other hand, since always $s(Q) \subseteq s^{rl}_Q(Q)$, we have $s(Q) = s^{rl}_Q(Q)$. Now let R^l_Q be an arbitrary left proper annihilator modulo Q. Then from $R^{lr}_Q = R + Q$ and $R \not\subset Q$ it follows that $R^l_Q \supset Q$. By the s-primarity of the ideal Q, $R^l_Q \subseteq s(Q) = s^{rl}_Q(Q)$, i.e. $s(Q)$ will be the unique maximal left proper annihilator modulo Q. Conversely, let $Q \neq K$ be such an ideal that $s(Q)$ is the unique maximal left proper annihilator modulo Q. Then $s(Q)$ will be the greatest left proper annihilator modulo Q, since every left annihilator modulo Q is contained in some maximal left proper annihilator modulo Q. But then from $B^r_Q \supset Q$ it follows that $B^{rl}_Q \subseteq s(Q)$, and since always $B \subseteq B^{rl}_Q$, we have $B \subseteq s(Q)$. Consequently, Q will be an s-primary ideal.

Corollary 1. Let s_i, $i = 1, 2$, be two arbitrary s-primarities. If an ideal $Q \neq K$ is both an s_1-primary and an s_2-primary ideal, then

$$s_1(Q) = \pi(Q) = s_2(Q).$$

Indeed, by Propositions 3 and 1, the ideals $s_1(Q)$, $s_2(Q)$, and $\pi(Q)$ are the unique maximal left annihilator modulo Q.

Denote by $\mathfrak{P}$ the set of all prime ideals of the ring K, and by $\mathfrak{P}_l$ the set of all its primal ideals. If S is an arbitrary s-primarity, then the set of all s-primary ideals of the ring K will be denoted by $\mathfrak{I}(s)$. By Proposition 1, $\mathfrak{P} \subseteq \mathfrak{I}(s) \subseteq \mathfrak{P}_l$.

Proposition 4. Let $\mathfrak{I}$ be a set of ideals of the ring K such that $\mathfrak{P} \subseteq \mathfrak{I} \subseteq \mathfrak{P}_l$. Then $\mathfrak{I} = \mathfrak{I}(s)$ for a suitable s-primarity S.

Proof. For $Q \neq K$, $Q \in \mathfrak{I}$, put $s(Q) = \pi(Q) = R^l_Q$, and if $Q \notin \mathfrak{I}$, put $s(Q) = p(Q) = Q$. It is verified that the constructed s-primarity will be the required one, i.e. the s-primary ideals are precisely the ideals in the set $\mathfrak{I}$.

A representation

$$Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n$$

of an ideal Q as the intersection of a finite number of ideals Q_i will be called an S-representation if S is an s-primarity and all Q_i are s-primary ideals. An S-representation (1) will be called s-reduced if no ideal Q_i can be omitted and the prime ideals $s(Q_i)$ are pairwise distinct.

In what follows we shall consider s-primarities S satisfying some of the following requirements:

$S1$. Every ideal has at least one S-representation.

П1. The intersection of a finite number of s-primary ideals belonging to one and the same prime ideal is again an s-primary ideal belonging to the same prime ideal.

$E1$. Every s-primary ideal Q has a unique s-reduced S-representation $(Q = Q)$.

Recall that an ideal is called irreducible if it is not a finite intersection of ideals strictly containing it. In the ring K every ideal is a finite intersection of irreducible ideals. Every

a prime ideal is irreducible and every irreducible ideal is primal.

Proposition 5. *An s-primality S satisfies condition S1 if and only if every irreducible ideal is s-primary. Among the s-primalities S satisfying condition S1, there exists a strongest one—the n-primality N. It is equivalent to irreducibility.*

Proof. Since the set $\mathfrak{N}$ of all irreducible ideals satisfies the inclusions $\mathfrak{P} \subseteq \mathfrak{N} \subseteq \mathfrak{P}_l$, by Proposition 4, $\mathfrak{N} = \mathfrak{I}(n)$, where n is some n-primality. Clearly, n-primality satisfies condition S1. Now let some s-primality S satisfy condition S1. Then, in particular, every irreducible ideal Q can be represented as the intersection of a finite number of s-primary ideals. Hence, by the definition of irreducible ideals, it follows that Q is an s-primary ideal.

Recall that an ideal Q is called **tertiary** ([6]) if from $AQ^r \supset Q$ and $AQ^r \cap B = Q$ it follows that $B = Q$ (A and B are ideals in K). For every ideal Q there exists an ideal $\mathrm{ter}(Q)$, which is maximal (and largest!) in the set of those ideals A

for which from $A^r \cap B = Q$ it follows that $B = Q$. Clearly, for every ideal Q, $K\,\mathrm{ter}(Q) \supset Q$ and $[\mathrm{ter}(Q)]Q^r \supset Q$. Therefore tertiarity is ter-primality.

Lemma (see ([6]), property 8.2). *If Q' and Q'' are tertiary ideals belonging to the prime ideals $\mathrm{ter}(Q')$ and $\mathrm{ter}(Q'')$, with $\mathrm{ter}(Q') \neq \mathrm{ter}(Q'')$, then from the equality $Q = Q' \cap X'' = Q'' \cap X'$, where X' and X'' are ideals, it follows that $Q = X' \cap X''$.*

Proposition 6. *Among the s-primalities S satisfying conditions S1 and P1, there exists a strongest s-primality. It is equivalent to tertiarity.*

Proof. Denote by $\mathfrak{Z}$ the set of all ideals representable as the intersection of a finite number of irreducible ideals belonging to one and the same prime ideal. Then $\mathfrak{P} \subseteq \mathfrak{N} \subseteq \mathfrak{Z} \subseteq \mathfrak{P}_l$. According to Proposition 4, there exists a t-primality T such that $\mathfrak{I}(t) = \mathfrak{Z}$. The constructed t-primality T satisfies conditions S1 and P1. Now let an s-primality S satisfy conditions S1 and P1. Then, by Proposition 5, we get $\mathfrak{Z} = \mathfrak{I}(t) \subseteq \mathfrak{I}(s)$. But this means precisely that the constructed t-primality T is the strongest among the s-primalities S satisfying conditions S1 and P1. Let us now prove that t-primality is equivalent to tertiarity. It is known (see ([6]), Theorem 8.2 and property 7.10) that tertiarity satisfies conditions S1 and P1. Since t-primality is the strongest s-primality satisfying these conditions, every t-primary ideal will be tertiary. Now let $Q \neq K$ be an arbitrary tertiary ideal. Represent Q as the intersection of a finite number of irreducible ideals

$$Q = Q_1 \cap Q_2 \cap \dots \cap Q_n.$$

Removing superfluous Q_i and combining the Q_i with one and the same prime ideal $s(Q_i)$, we obtain a decomposition

$$Q = Q_1' \cap Q_2' \cap \dots \cap Q_m',$$

where the Q_i' are t-primary ideals, and none of the Q_i' can be removed. If $m = 1$, then everything is proved. Now let $m > 1$. Since all Q_i' are at the same time tertiary ideals, by the lemma, from the equalities

$$Q = Q_1' \cap Q_2' \cap \dots \cap Q_m' = Q \cap K$$

there follows the existence of such an i that

$$Q = Q_1' \cap \dots \cap Q_{i-1}' \cap Q_{i+1}' \cap \dots \cap Q_m' \cap K = Q_1' \cap \dots \cap Q_{i-1}' \cap Q_i' \cap \dots \cap Q_m'.$$

We have obtained that Q_i' is a superfluous ideal—a contradiction. The proposition is proved.

Corollary 2. *An ideal of a ring is tertiary if and only if it is representable as the intersection of a finite number of irreducible ideals belonging to one and the same prime ideal.*

Uniqueness theorem for tertiarity. *In order that an s-primality satisfy conditions S1, P1, and E1, it is necessary and sufficient that it be equivalent to tertiarity.*

Proof. As is known ([6]), tertiariness satisfies the requirements S1, P1, and E1. Suppose now that some s-primarity satisfies these same conditions. We shall show that every s-primary ideal Q is tertiary. By condition S1, Q can be represented in the form of a finite intersection (1) of irreducible ideals Q_i. As in the proof of Proposition 6, one may assume that none of the Q_i can be omitted and that all Q_i are t-primary ideals, and hence also s-primary. Therefore the representation (1) will be s-reduced and, by requirement E1,

$$Q = Q_1 = Q_2 = \cdots = Q_n,$$

i.e. Q is a t-primary ideal. According to Proposition 6, Q will be a tertiary ideal. Conversely, by the same Proposition 6, every tertiary ideal Q will be s-primary.

Recall that an ideal Q is called **primary** if $Bq' \supset Q$ implies $B \subseteq L(Q)$, where $L(Q)$ is the largest nilpotent ideal modulo Q, or, equivalently, if $BA \subseteq Q$ and $A \nsubseteq Q$ imply $B^n \subseteq Q$ ([7]). Since primarity in the case of a Noetherian ring satisfies the conditions S1, P1, and E1 ([7]), it is equivalent to tertiariness. Thus, as a corollary of the assertions proved above, we obtain the following proposition:

Uniqueness theorem for primarity. *In order that, in a Noetherian ring, s-primarity satisfy the requirements S1, P1, and E1, it is necessary and sufficient that it be equivalent to classical primarity. The primary ideals are precisely those ideals representable as the intersection of a finite number of irreducible ideals belonging to one and the same prime ideal.*

We note that the results obtained are valid for Noetherian $\mathfrak{C}$-algebras ([6]), and therefore are applicable to semigroups and modules. In view of the importance of the concept of a tertiary ideal, it is of interest to study tertiary rings, i.e. rings whose zero ideal is tertiary. Tertiary rings are a special case of connected rings, which were considered in the note ([8]).

Institute of Mathematics with Computing Center
Academy of Sciences of the MSSR

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Note: Figure translations are in progress. See original paper for figures.