CRITICISM AND BIBLIOGRAPHY

G. N. Polozhii. EQUATIONS OF MATHEMATICAL PHYSICS.

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Abstract

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G. N. Polozhii. EQUATIONS OF MATHEMATICAL PHYSICS. PUBLISHING HOUSE "VYSSHAYA SHKOLA," Moscow, 1964, 559 pp.

The book, approved by the Ministry of Higher and Secondary Specialized Education of the USSR as a textbook for students of mechanics-and-mathematics and physics-and-mathematics faculties of universities, was written by the author, as noted in the preface, on the basis of a revised and supplemented course of lectures that he delivered over a number of years at the Mechanics and Mathematics Faculty of Kiev State University. The order adopted in it for presenting the material differs from the traditional one used in other existing textbooks on this discipline. Thus, for example, the treatment of parabolic and hyperbolic partial differential equations is preceded by a detailed study of the general properties of elliptic equations and of the mathematical apparatus associated with them. Separate chapters are devoted to questions connected with the consideration of physical processes leading to the fundamental equations of mathematical physics, as well as to questions belonging to the general theory of partial differential equations. All the material is presented mainly from the point of view of classical mathematical physics. A characteristic feature is that the basis of all possible methods for solving and investigating the principal boundary-value problems posed for equations of elliptic, parabolic, and hyperbolic types is the broad use of all kinds of delta-shaped functions and formulas for transforming n-dimensional integrals into $(n-1)$-dimensional ones. The method of separation of variables and methods of integral transforms are considered as applied simultaneously to all three types of equations.

The contents of the book by chapters are as follows.

In the first chapter, which is essentially introductory in character, the basic definitions are given and a number of the simplest problems from various areas of physics and mechanics are considered, leading to the fundamental equations of mathematical physics.

In the second chapter some general questions of the theory of partial differential equations are set forth, considered from a purely mathematical point of view. Here, after introducing the concept of a normal system of partial differential equations, a complete proof is given of S. Kovalevskaya' s theorem on the existence and uniqueness of the solution of the Cauchy problem for such a system in the class of analytic functions. A solution of the Cauchy problem is also given for a general quasilinear system of equations. The concept of characteristics, closely connected with this problem, is given as a derivative of the concept of characteristic variables (a characteristic variable is any function $\varphi(x_1, \dots, x_n)$

satisfying the equation

$$\left| \sum_{k_1+\dots+k_n=r_j} A_{ij}^{k_1\dots k_n}\, \varphi_{x_1}^{k_1} \cdots \varphi_{x_n}^{k_n} \right| = 0; \quad i,j = 1,\dots,p,$$

provided that

$$\sum_{i=1}^{n} |\varphi_{x_i}|^2 \neq 0,$$

$A_{ij}^{k_1\dots k_n}(x_1,\dots,x_n)$ are the coefficients of the highest derivatives in the equations of the system). The classification of quasilinear systems of equations is considered in its most general form. Its basic concepts, explained by simple examples, are extended to the case of completely arbitrary systems of partial differential equations. Separately, the classification of second-order linear equations and their reduction to canonical form are considered. At the end of the chapter the solution of the Cauchy problem for first-order equations is given.

The third chapter is devoted to linear second-order equations of elliptic type. It begins with the formulation of the principal boundary-value problems for the Laplace and Poisson equations and with the physical interpretation of their content. The methods of their solution and investigation are based on the application of:

1) the functions of a unit source at the point P_0

$$\delta(P, P_0) = \frac{1}{4\pi r}$$

and the functions of a unit dipole at the point P_0

$$\mu(P, P_0) = \lim_{P_0' \to P_0} \frac{\delta(P, P_0') - \delta(P, P_0)}{\rho}$$

with axis $\vec{n}_0$ for the three-dimensional Laplace equation (here r is the distance between the points $P(x,y,z)$ and $P_0(x_0,y_0,z_0)$, ρ is the distance between the points P_0' and P, and the point P_0' tends to the point P along the rectilinear directed segment $\vec{n}_0$ issuing from P_0), and analogous functions for the two-dimensional Laplace equation:

$$\delta(z, z_0) = \frac{1}{2\pi} \ln \frac{1}{r}, \qquad \mu(z, z_0) = \lim_{z_0' \to z_0} \frac{\delta(z, z_0') - \delta(z, z_0)}{\rho},$$

where $r = |z - z_0|$, $z = x + iy$, $z_0 = x_0 + iy_0$, and ρ is the distance between the points z_0' and z_0; the point z_0' tends to the point z_0 along the rectilinear directed segment $\vec{n}_0$ issuing from the point z_0, and

2) the Ostrogradsky (Green) formulas, which transform volume integrals (area integrals) into surface integrals (contour integrals). Here the concept of the influence function (Green' s function) of the corresponding boundary-value problem is introduced, and an integral representation of their solutions through the influence function is given. Cases are indicated in which influence functions can be effectively constructed by the method of sources, and explicit solutions of boundary-value problems are presented for the sphere, circle, and certain other plane regions. After the introduction of the general definition of harmonic functions of three and two variables, their general properties are considered, and examples are given of their application to the study of the principal boundary-value problems for the Laplace equation. In accordance with the work of A. M. Lyapunov on potential theory, general questions of the theory of the potentials of a simple layer, double layer, and volume (area) are set forth, the results of which are then applied to the solution and study of the principal boundary-value problems of potential theory by the method of Fredholm integral equations (the interior and exterior Dirichlet and Neumann problems). The theory of the potential for the equation $\Delta u - k^2 u = 0$ is considered separately; the properties of its solutions, as is well known, are in many respects analogous to the properties of harmonic functions. Integral representations for its solutions in the case of three and two independent variables are derived. In conclusion, the chapter indicates the fundamental distinctions by which the equation $\Delta u - k^2 u = 0$ and the Laplace equation differ from the Helmholtz equation $\Delta u + k^2 u = 0$, both from the point of view of the general properties of their solutions and from the point of view of the formulation and uniqueness of the solution of boundary-value problems for them.

The fourth chapter is devoted to linear second-order equations of parabolic type. Like the preceding one, it begins with the formulation of the principal and most frequently encountered boundary-value problems for the heat-conduction equation and the physical interpretation of their content. Here their generalizations are also considered—the first generalized boundary-value problem (the first boundary-value problem with a boundary depending on time t) and the generalized mixed problem (the mixed boundary-value problem for a region with variable boundaries). The methods for solving and studying boundary-value problems are based on the use of:

1) the functions of a unit instantaneous source at the point P' at the time t'

$$\delta(P, P', t - t') = \left(\frac{1}{2a\sqrt{\pi(t - t')}} \right)^n e^{-\frac{r^2}{4a^2(t-t')}}$$

for the heat-conduction equation (here $n = 1, 2$ or 3, depending on the case under consideration, r is the distance between the points P and P', $t > t'$) and the functions of a unit instantaneous dipole at the point P' at the time t'

$$\mu(P, P', t - t') = \frac{\partial}{\partial n'} \delta(P, P', t - t')$$

with axis $\vec{n}'$ ($\vec{n}'$ is the directed segment issuing from the point P'), which, as is known, in the study of the heat-conduction equation play the same role as the func-

···the unit source and the unit dipole in potential theory for elliptic equations, and

2) formulas for transforming n-fold integrals into $(n-1)$-fold integrals.

Integral representations of solutions of the first, second, and third boundary-value problems for the heat-conduction equation are derived in terms of the corresponding influence functions (Green's functions), which are then used to study the differential properties of the solutions. Examples are given of the explicit construction of influence functions for particular domains (an n-dimensional half-space, an n-dimensional layer for the first and second boundary-value problems, and an n-dimensional half-space for the third boundary-value problem). The δ-function method is explained. Thermal potentials of a simple and a double layer are studied, on the basis of which the basic boundary-value problems for the heat-conduction equation are then solved by the method of Volterra integral equations. In conclusion of the chapter, generalized thermal potentials of a simple and a double layer are considered; their properties are studied in detail for the one-dimensional case. The results of this investigation are applied to the solution of the generalized first and generalized mixed boundary-value problems by the method of Volterra integral equations.

In the fifth chapter, linear equations of the second order of hyperbolic type are considered. At the beginning of the chapter the first, second, and third boundary-value problems (and their generalizations) and the Cauchy problem for the wave equation are formulated. Their solution and investigation are based on the use of

1) the unit impulse function and the Riemann function,

2) formulas for transforming n-fold integrals into $(n-1)$-fold integrals.

Here, just as in the case of equations of elliptic and parabolic types, unlike other well-known textbooks on equations of mathematical physics, the concept of the influence function (Green's function) is introduced, and formulas are constructed that give integral representations of the solutions of boundary-value problems for various domains in terms of influence functions. The Cauchy problem for the equation of string vibrations is solved by d' Alembert' s method. A solution is given of the Cauchy problem for the three-dimensional (two-dimensional) wave equation, presented for the general case as a sum of Poisson’ s integral and a retarded potential. The physical conclusions following from the solution of the Cauchy problem for the homogeneous wave equation are drawn. Among the main problems considered in this chapter are also the so-called characteristic problems (with conditions on characteristics), as well as problems connected with steady-state oscillations and problems connected with wave propagation. Much attention is devoted to the latter investigations. Integral representations of solutions of the wave equation are given for the one-dimensional, two-dimensional, and three-dimensional cases. By analogy with the potentials for elliptic and parabolic equations, wave potentials are introduced—retarded simple-layer potentials, retarded double-layer potentials, and retarded volume potentials. The problem is considered of studying, as a limiting case, the Cauchy problem for the wave equation. By the method of added variable, the Cauchy problem for the one-dimensional and two-dimensional telegraph equations is solved. Explicit formulas are given that solve the Cauchy problem for the three-dimensional telegraph equation, and also for the n-dimensional wave equation when $n > 3$. A method is presented for solving the Cauchy problem by means of Ostrogradsky’ s formula for adjoint differential operators and the Riemann function. Here, too, examples are given of the construction of the Riemann function for the telegraph equation and Einstein’ s equation—Poisson’ s equation. In problems on steady-state oscillations, much attention is devoted to the consideration of conditions ensuring uniqueness of the desired solution (radiation conditions). In conclusion of the chapter, the equation of propagation of discontinuities of solutions and generalized solutions is considered; the problem of the propagation of a wave front is solved, as are certain questions of wave dispersion. Using the example of the Cauchy problem for the simplest linear system of first-order hyperbolic equations, one of the approximate methods for solving hyperbolic equations is illustrated—the method of computation along characteristics.

The sixth chapter is devoted to the method of separation of variables and its application to solving boundary-value problems connected with trigonometric functions of a multiple argument for equations of all three types. Here solutions are given of the first boundary-value problem for the one-dimensional wave equation, for the one-dimensional heat-conduction equation, the Dirichlet problem for a circle and a rectangle, and the problem of oscillations of a rectangular membrane. A general theory of the one-dimensional problem on eigenvalues and eigenfunctions is set forth, which is then applied to the justification of the method of separation of variables. After the definition of adjoint and self-adjoint boundary-value problems, the properties of the eigenvalues and eigenfunctions of the one-dimensional Sturm–Liouville problem are studied. In particular, proofs are given of V. A. Steklov’ s theorem on the expansion of a function in eigenfunctions of the Sturm–Liouville problem and of the theorem on the completeness of the system of eigenfunctions. Asymptotic formulas are derived for eigenvalues, eigenfunctions, and their derivatives. With the aid of the so-called accompanying boundary conditions introduced, special theorems are established on Fourier coefficients in expansions in eigenfunctions of the···

the Sturm–Liouville problem is given, and their application to the substantiation of the method of separation of variables for obtaining solutions in the classical sense is presented. Using the example of a boundary-value problem for parabolic and hyperbolic equations of general form with two independent variables, the method of separation of variables is also considered from the point of view of generalized solutions. The question of the approximate determination of eigenvalues and eigenfunctions is discussed, and their variational properties are studied, making it possible to use the Ritz method for such purposes. The chapter concludes with a discussion of the simplest properties of cylindrical and spherical functions from the standpoint of their application to the solution of boundary-value problems by the method of separation of variables.

The seventh chapter is devoted to methods of integral transforms. Here the Fourier integral method is considered, using as an example its application to the solution of the Cauchy problem for the wave equation; the Laplace transform and methods of finite integral transforms are treated: the Fourier–Bessel integral, the Mellin transform, and the Laplace transform. The basic (simplest) properties of operational calculus are illustrated by the example of the Laplace–Carson transform.

As the author himself notes in the preface to the book, the requirement that its length be limited did not allow him to cover with equal completeness the entire range of questions belonging to the equations of mathematical physics. However, this shortcoming of the book is partly remedied by the presence in it of detailed references to special journal literature and monographs, in which the reader may find a detailed exposition of individual questions that are not covered sufficiently fully in the book.

On the whole, the book is distinguished by the fact that, despite its comparatively small size, it contains a large amount of factual material and is written in such a way that, in order to understand its main text, there is no need to consult any additional literary sources. In the treatment of individual questions the book contains a number of improvements and simplifications obtained either on the basis of specialized journal and monographic literature or through the author’ s original constructions.

The clear and generally consistent presentation of the material, and the large number of problems for equations of mathematical physics considered in the book, whose solutions can be obtained explicitly, make it, in our view, very useful not only for university students, but also for graduate students, researchers at scientific institutions, and engineers studying questions of physics and mechanics connected with partial differential equations.

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Note: Figure translations are in progress. See original paper for figures.