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Abstract

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Yu. S. NIKOL' SKII

ON THE THEORY OF WEIGHT CLASSES OF DIFFERENTIABLE FUNCTIONS OF SEVERAL VARIABLES AND ITS APPLICATIONS TO BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC EQUATIONS

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The present investigations are close to the works of L. D. Kudryavtsev on weight classes of differentiable functions of several variables in unbounded domains; among these works we mention ([1–3]).

Let E_n be the n-dimensional space of points $x = (x_1, \dots, x_n)$ with real coordinates, and let Ω_1 be a bounded domain containing the origin $O = (0, \dots, 0)$, whose boundary is an $(n-1)$-dimensional surface Γ of class $C^{(r)}$. Let Ω be the set of all points x lying outside Γ. We define on $\overline{\Omega}$ a continuous function $\varphi = \varphi(\rho)$ $\left(\rho = \sqrt{x_1^2 + \dots + x_n^2}\right)$ such that, for some positive d, the inequality $\varphi(\rho) \geq d$ holds for all $x \in \overline{\Omega}$. We consider the class $W^{(r)}_{p,\varphi}(\Omega)$ of functions $f = f(x)$, defined on Ω together with their generalized derivatives up to order r inclusive, for which the finite norm

$$\|f\|_{W^{(r)}_{p,\varphi}(\Omega)} = \|f\|_{L_p(\Omega^*)} + \sum_{|k|=r} \left\| \frac{f^{(k)}(x)}{\varphi(\rho)} \right\|_{L_p(\Omega)}$$

has meaning, where r is a nonnegative integer; $1 < p < \infty$; Ω^* is the part of Ω belonging to a ball with center at the origin and containing Γ strictly inside it. As usual, we assume that

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p \, d\Omega \right)^{1/p}, \qquad f^{(k)}(x) = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}},$$

where $k = (k_1, \dots, k_n)$ is an integer vector ($k_j \geq 0$, $j = 1, 2, \dots, n$), $|k| = k_1 + \dots + k_n$.

We restrict ourselves to the consideration of functions φ representable in the form $\varphi = \rho^{\alpha_0} \lambda(\rho)$ $(\alpha_0 = (n-p)/p)$, where

$$\int_a^\infty \frac{dz}{z\varkappa(z)} < \infty, \qquad \varkappa(z) = \left\{ \min_{a\le\rho<\infty} \left[\frac{\lambda(z\rho)}{\lambda(\rho)}\right]^p \right\}^{1/p}, \qquad a = \max_{x\in\Gamma} \rho. \tag{1}$$

For such classes we prove the validity of the following embeddings*:

Theorem 1. The following embeddings hold:

$$W^{(r)}_{p,\varphi}(\Omega) \to W^{(r-k)k}_{p,\varphi\rho^k}(\Omega), \qquad k = 0, 1, \dots, r; \tag{2}$$

$$W^{(r)}_{p,\varphi}(\Omega) \to W^{(r-k)}_{p,\varphi\rho^{k(1-1/p)}}(\Lambda_{n-k}), \tag{3}$$

where $k = 0, 1, \dots, r,\ k < n,\ \Lambda_{n-k} = \Omega \cdot E_{n-k}$.

* As is customary, we assume that if B_1, B_2 are normed spaces, then $B_1 \to B_2$ means that $B_1 \subseteq B_2$ and there exists a constant $c > 0$, independent of $x \in B_1$, such that $\|x\|_{B_2} \le c\|x\|_{B_1}$.

We note that if $\lambda(\rho) = \rho^\varepsilon (\varepsilon > 0)$, the embedding (2) and the embedding (3) for $k = 1$ were proved by L. D. Kudryavtsev ([1]). From condition (1) it is clear that the growth of the function φ at infinity in such theorems may be arbitrarily large, provided only that it is bounded below by a certain condition. Condition (1) is sufficient for the validity of the indicated embeddings; however, one can give examples of such weight functions φ for which the integral (1) diverges and at the same time the embeddings (2), (3) no longer hold.

Consider the variational problem in which embedding (2) is applied. Let

$$E(f,g) = \int_\Omega \sum_{|k|,|l|\le r} a_{kl}(x) f^{(k)}(x) g^{(l)}(x)\, d\Omega, \qquad E(f) = E(f,f),$$

where the sum is extended over all possible pairs of integer nonnegative vectors $k = (k_1, \dots, k_n)$, $l = (l_1, \dots, l_n)$, for which $|k|, |l| \le r$, and $a_{kl}(x) = a_{lk}(x)$ are functions measurable on Ω, for which the inequalities

$$|a_{kl}(x)| \le M^2/[\rho^{r-\min(|k|,|l|)}\varphi(\rho)]^2$$

hold.

It is assumed that there exists a number $\lambda > 0$, independent of $x \in \Omega$, such that

$$\sum_{|k|,|l|\le r} a_{kl}(x)\xi_k\xi_l \ge \frac{\lambda}{(\varphi(\rho))^2} \sum_{|k|=r} \xi_k^2,$$

where ξ_k are variables corresponding to the vectors k. On the boundary Γ of the domain Ω, prescribe functions $\psi_s \in W_2^{2(r-s-1/2)}(\Gamma)$ $(s = 0, 1, \dots, r-1)$, where $W_2^{(\alpha)}(\Gamma)$ (α non-integer) denote the usual fractional Sobolev classes (see, for example, ([4])). Let $\mathfrak{M}$ be the class of functions $f \in W_{2,\varphi}^{(r)}(\Omega)$ satisfying the boundary conditions

$$\partial^s f/\partial n^s\big|_\Gamma = \psi_s, \qquad s = 0, 1, \dots, r-1.$$

In the paper it is proved that the class $\mathfrak{M}$ is nonempty.

Let us also prescribe on Ω a function $F = F(x)$ possessing the property that

$$|(F, v)| = \left|\int_\Omega F(x)v(x)\,d\Omega\right| \le c_F\|v\|_{W_{2,\varphi}^{(r)}(\Omega)}, \tag{4}$$

where c_F is a constant depending on F, but not depending on $v \in W_{2,\varphi}^{(r)}(\Omega)$. Obviously, if

$$\int_\Omega \rho^{2r}\varphi^2 F^2\,d\Omega < \infty,$$

then condition (4) is fulfilled. In the paper it is shown what more precise sufficient condition the function F must satisfy in order that condition (4) hold. Consider also the functional

$$K(f) = E(f) - 2(F, f).$$

Under the assumptions made, we prove the following theorems.

Theorem 2. *In the class $\mathfrak{M}$ the functional $K(f)$ is bounded from below; moreover, there exists, and is unique, a function $u \in \mathfrak{M}$ giving the minimum of the functional $K(f)$ in the class $\mathfrak{M}$.*

This theorem is equivalent to the following theorem.

Theorem 3. *In the class $\mathfrak{M}$ there exists a unique generalized solution of the equation*

$$\sum_{|k|,|l|\le r} (-1)^{|l|}\frac{\partial^l}{\partial x^l}\left(a_{kl}(x)u^{(k)}(x)\right) = F(x), \tag{5}$$

which is the Euler equation of the functional $K(f)$.

Theorem 4. There exists a constant $c > 0$ such that the generalized solution u of the boundary-value problem under consideration (in Ω) satisfies the inequality

$$\|u\|_{W^{(r)}_{2,\varphi}(\Omega)} \leq c \left\{ \sum_{s=0}^{r-1} \|\psi_s\|_{W^{(r-s-1/2)}2(\Gamma)} + c_F \right\},$$

where

$$c_F = \sup_{\|v\|_{W^{(r)}_{2,\varphi}(\Omega)} \leq 1} |(F, v)|.$$

If one requires that the functions $a_{kl}(x)$ and $F(x)$ be sufficiently smooth, then it follows from known results (see, for example, ([5–8])) that the solution u has continuous partial derivatives on Ω up to order $2r$ inclusive, and u becomes a classical solution of equation (5).

Theorem 5. The classical solution of equation (5) in the class $\mathfrak{M}$ is unique.

The proof of the last theorem is based essentially on the fact that it is shown that the class $\mathfrak{M}_{00}$ is everywhere dense in the class $\mathfrak{M}_0$ in the sense of the metric

$$D_\Omega(f) = \int_\Omega \sum_{|k|=r} \left[\frac{f^{(k)}(x)}{\varphi(\rho)} \right]^2 d\Omega,$$

where $\mathfrak{M}_0$ is the class of functions $f \in W^{(r)}_{2,\varphi}(\Omega)$ having zero boundary values on Γ ($\psi_s = 0,\ s = 0, 1, \dots, r-1$), and $\mathfrak{M}_{00}$ is the class of finite functions $f \in \mathfrak{M}_0$.

We note that these investigations develop the works of L. D. Kudryavtsev ([2,3]), in which the variational method for solving the first boundary-value problem was considered, in the case of an unbounded domain, for self-adjoint elliptic equations of second order in a somewhat different formulation.

In conclusion I express my deep gratitude to L. D. Kudryavtsev for formulating the problem and for his constant attention.

Moscow Institute of Physics and Technology

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CITED LITERATURE

[1] L. D. Kudryavtsev, DAN, **153**, No. 3, 530 (1963).
[2] L. D. Kudryavtsev, *On the first boundary-value problem for elliptic equations with coefficients decreasing at infinity*, Soviet-American Symposium on Partial Differential Equations, Novosibirsk, 1963.
[3] L. D. Kudryavtsev, DAN, **157**, No. 1, 45 (1964).
[4] L. I. Slobodetskii, DAN, **118**, No. 2, 243 (1958).
[5] P. I. Lizorkin, DAN, **134**, No. 4, 761 (1960).

[6] O. A. Ladyzhenskaya, DAN, **79**, No. 5, 723 (1951).
[7] O. A. Ladyzhenskaya, DAN, **120**, No. 5, 956 (1958).
[8] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates for solutions of elliptic equations near the boundary under general boundary conditions*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.