A. F. Shadrov

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Abstract

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A. F. Shadrov

ESTIMATES OF HARTOGS COEFFICIENTS OF FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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In papers ($^{1-10}$) estimates were obtained for Taylor coefficients for certain classes of functions of two and of a larger number of complex variables. In the present note analogous estimates are established for Hartogs coefficients of the corresponding functions, regular in multiple Hartogs domains.

For brevity of notation we introduce the designations:

$$w = (z_1, \dots, z_m), \quad z = (z_{m+1}, \dots, z_n), \quad k = (k_1, \dots, k_m),$$

$$\|k\| = k_1 + \dots + k_m, \quad w^k = z_1^{k_1} \cdots z_m^{k_m}, \quad |w|^k = |z_1|^{k_1} \cdots |z_m|^{k_m}.$$

Let

$$D\{|z_j| < R_j(z),\ j = 1, \dots, m;\ z \in H_D\}$$

be an arbitrary bounded complete m-fold Hartogs domain with planes of symmetry $z_1 = 0, \dots, z_m = 0$ over the space C^n of n complex variables $z_1, \dots, z_n$ (see (^{11,12})). It is known (^{11,12}) that a function $F(w, z)$, regular in the domain D, is representable in D by an m-fold Hartogs series

$$F(w, z) = \sum_{k=0}^{\infty} f_k(z) w^k.$$

Theorem 1. If in the domain D the function

$$F(w, z) = \sum_{k=0}^{\infty} f_k(z) w^k,$$

where $f_0(z)$ is given, is regular and $|F(w, z)| \leq 1$, then for $\|k\| > 0$

$$|f_k(z)| \leq (1 - |f_0(z)|^2) d_k^{-1}, \tag{1}$$

where

$$d_k = \sup_{(w,z) \in D} (|w|^k);$$

equality for $k_1 > 0, \dots, k_m > 0$ occurs only for the function

$$F(w, z) = \frac{d_k f_0(z) + \eta_k(z) w^k}{d_k + \overline{f_0(z)}\, \eta_k(z) w^k}, \quad |\eta_k(z)| = 1. \tag{2}$$

Proof. Take an arbitrary point $z_0 \in H_D$. The function

$$F(w, z_0) = \sum_{k=0}^{\infty} f_k(z_0) w^k$$

is regular in the polydisc $E\{|z_j| < R_j(z_0),\ j = 1, \dots, m;\ z = z_0\}$ and, by the hypothesis of the theorem, $|F(w, z_0)| \leq 1$. By theorem 4 (^{9}), for $\|k\| > 0$ we have

$$|f_k(z_0)| \leq (1 - |f_0(z_0)|^2)\, R_1^{-k_1}(z_0) \cdots R_m^{-k_m}(z_0). \tag{3}$$

Passing on the right-hand side to the lower bound over all polydiscs $E \subset D$, we obtain the estimate (1).

By theorem 4.1 (^{9}), the equality sign in estimate (3) occurs only for the function

$$F(w, z_0) = \frac{f_0(z_0) R_1^{k_1}(z_0) \cdots R_m^{k_m}(z_0) + \eta_k(z_0) w^k}{R_1^{k_1}(z_0) \cdots R_m^{k_m}(z_0) + \overline{f_0(z_0)}\, \eta_k(z_0) w^k}, \quad |\eta_k(z_0)| = 1.$$

Since z_0 is an arbitrary fixed point of H_D, equality in estimate (1) for $k_1 > 0, \dots, k_m > 0$ occurs only for the function (2).

Corollary. If in the domain D the function

$$F(w, z) = \sum_{k=0}^{\infty} f_k(z) w^k$$

is regular and $|F(w, z)| \leq 1$, then for $\|k\| > 0$

$$|f_k(z)| \leq d_k^{-1}; \tag{4}$$

equality for $k_1 = 0, \dots, k_m = 0$ and $k_1 > 0, \dots, k_m > 0$ occurs only for the function

$$F(w, z) = \eta_k(z) w^k d_k^{-1}, \qquad |\eta_k(z)| = 1. \tag{5}$$

Proof. The estimate $|f_0(z)| \leq 1$ is obvious. By the theorem, for $\|k\| > 0$ we have

$$|f_k(z)| \leq (1 - |f_0(z)|^2) d_k^{-1}; \tag{6}$$

since

$$(1 - |f_0(z)|^2) d_k^{-1} \leq d_k^{-1}, \tag{7}$$

we also have estimate (4) for $k_1 > 0, \dots, k_m > 0$.

The equality sign in estimate (6), by Theorem 1, occurs only for the function (2). And since equality in (7) occurs only when $f_0(z) = 0$, equality in estimate (4) for $\|k\| > 0$ occurs only for the function (5).

Theorem 2. If the function

$$F(w, z) = \sum_{k=0}^{\infty} f_k(z) w^k \qquad (f_0(z) = 0)$$

is regular in the domain D and for any (w', z') and (w'', z'') in D satisfies the condition

$$F(w', z')F(w'', z'') \neq 1,$$

then for $\|k\| > 0$

$$|f_k(z)| \leq d_k^{-1};$$

equality for $k_1 > 0, \ldots, k_m > 0$ occurs only for the function

$$F(w, z) = \eta_k(z) w^k d_k^{-1}.$$

Theorem 3. If the function

$$F(w, z) = \sum_{k=0}^{\infty} f_k(z) w^k,$$

regular in the domain D, satisfies in D the condition

$$\operatorname{Re} F(w, z) \leq U,$$

then for $\|k\| > 0$

$$|f_k(z)| \leq 2(U - \operatorname{Re} F(0, z)) d_k^{-1}. \tag{8}$$

Theorems 2 and 3 are proved analogously to Theorem 1, with the aid of Theorems 4.2 and 4.3, respectively (^{9}).

Remark. Estimate (8) is sharp, since it is attained by the function

$$F(w, z) = U + i \operatorname{Im} f_0(z) + (U - \operatorname{Re} f_0(z)) \frac{\eta_k(z) w^k + d_k}{\eta_k(z) w^k - d_k}, \qquad |\eta_k(z)| = 1$$

(U is real and $\operatorname{Re} f_0(z) < U$).

Corollary 1. If the function

$$F(w, z) = \sum_{k=0}^{\infty} f_k(z) w^k$$

is regular in the domain D and satisfies in D the condition $\operatorname{Re} F(w, z) \geq 0$, then for $\|k\| > 0$

$$|f_k(z)| \leq 2 \operatorname{Re} f_0(z) d_k^{-1}. \tag{9}$$

Corollary 2. If the function

$$F(w, z) = \sum_{0}^{\infty} f_k(z) w^k \qquad (f_0(z) = 1),$$

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regular in the domain D, satisfies in D the condition $\operatorname{Re} F(w,z) \geqslant 0$, then for $\|k\| > 0$

$$|f_k(z)| \leqslant 2d_k^{-1}. \tag{10}$$

Corollary 3. If the function

$$F(w,z) = \sum_{k=0}^{\infty} f_k(z) w^k \quad (f_0(z) = 1)$$

is regular in the domain D and satisfies in D the condition $\operatorname{Re} F(w,z) \geqslant \delta$ $(0 \leqslant \delta \leqslant 1)$, then for $\|k\| > 0$

$$|f_k(z)| \leqslant 2(1-\delta) d_k^{-1}. \tag{11}$$

Corollary 4. If the function

$$F(w,z) = \sum_{k=0}^{\infty} f_k(z) w^k$$

is regular in the closed domain $\overline{D}$, then for $\|k\| > 0$

$$|f_k(z)| \leqslant 2(A_F - \operatorname{Re} F(0,z)) d_k^{-1}, \tag{12}$$

where

$$A_F = \max_{(w,z)\in D} \operatorname{Re} F(w,z).$$

Remark. The estimates (9)–(12) are sharp, since there exist functions for which they are attained.

Theorem 4. If in the closed domain $\overline{D}$ the function

$$F(w,z) = \sum_{k=0}^{\infty} f_k(z) w^k,$$

where $f_0(z)$ is given, is regular, then for $\|k\| > 0$

$$|f_k(z)| \leqslant (M - |f_0(z)|^2 M^{-1}) d_k^{-1}, \tag{13}$$

where

$$M = \max_{(w,z)\in\overline{D}} |F(w,z)|;$$

equality holds only for the function

$$F(w,z) = M\,\frac{d_k f_0(z) + \eta_k(z) w^k}{d_k + \overline{f_0(z)}\,\eta_k(z) w^k}, \qquad |\eta_k(z)| = 1 \tag{14}$$

(M is a positive constant).

Proof. Consider the function $\Phi(w,z) = F(w,z)/M$, $M = \max_{(w,z)\in\overline{D}} |F(w,z)|$. It is regular in the domain D and satisfies in D the condition $|\Phi(w,z)| \leqslant 1$; therefore, by Theorem 1, for $\Phi(w,z)$ we have

$$|f_k(z)/M| \leqslant (1 - |f_0(z)|^2 M^{-2}) d_k^{-1}. \tag{15}$$

Hence estimate (13) follows.

By Theorem 1, equality in estimate (15) for $\|k\| > 0$ occurs only for the function

$$\frac{F(w,z)}{M} = \frac{d_k f_0(z) + \eta_k(z) w^k}{d_k + \overline{f_0(z)}\,\eta_k(z) w^k}, \qquad |\eta_k(z)| = 1.$$

Hence it follows that equality in estimate (13) occurs only for function (14).

Corollary. If in the closed domain $\overline{D}$ the function

$$F(w,z) = \sum_{k=0}^{\infty} f_k(z) w^k$$

is regular, then for $\|k\| > 0$

$$|f_k(z)| \leqslant M d_k^{-1}, \qquad M = \max_{(w,z)\in\overline{D}} |F(w,z)|;$$

equality for $k_1 = 0, \dots, k_m = 0$ and $k_1 > 0, \dots, k_m > 0$ holds only for the function

$$F(w,z) = M \eta_k(z) w^k d_k^{-1}, \qquad |\eta_k(z)| = 1.$$

Theorem 5. If the function

$$F(w,z) = \sum_{k=0}^{\infty} f_k(z) w^k$$

regular in the domain D, is bounded in D, $|F(w,z)| < M$, and does not take the value zero in D, then for $\|k\| > 0$

$$|f_k(z)| \leqslant 2|f_0(z)|\, d_k^{-1} \ln \frac{M}{|f_0(z)|}.$$

The theorem is proved analogously to Theorem 1 with the aid of Theorem 3 (^{10}) for the case of a polycylinder.

Remark. For the domain $D^{(R)}\{|z_j| < R_j,\ j = 1, \dots, m;\ z \in H_D\}$, where R_j are positive constants, $d_k = R_1^{k_1} \cdots R_m^{k_m}$. In particular, for the domain $D^{(1)}\{|z_j| < 1,\ j = 1, \dots, m;\ z \in H_D\}$ we have $d_k = 1$. For these domains all the estimates considered take a more special form.

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Note: Figure translations are in progress. See original paper for figures.

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