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# Mathematics

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1965

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**Abstract**

**Full Text**

**Mathematics**

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## ON REAL ENTIRE FUNCTIONS BELONGING TO THE CLASS $W_\sigma^{(1)}$

*(Presented by Academician S. N. Bernstein on 31 X 1964)*

Denote by  $W_\sigma^{(1)}$  the class of real entire functions  $\varphi_\sigma(z) = \sum_{k=0}^{\infty} c_k z^k$  of finite degree  $\sigma$ , nonnegative on the real axis and such that

$$A_{\varphi_\sigma} \equiv \|\varphi_\sigma\| = \int_{-\infty}^{\infty} \varphi_\sigma(x) dx < \infty. \quad (1)$$

Let  $F_j(\varphi_\sigma)$  ( $j = 1, \dots, s$ ) be linear functionals defined on the set  $W_\sigma$ . We shall call a function  $\varphi_\sigma^*(x) \in W_\sigma^{(1)}$  an **extremal function** of the problem if its norm  $A_{\varphi_\sigma}$  assumes the smallest value among the norms of all functions  $\varphi_\sigma(x) \in W_\sigma^{(1)}$  subject to the constraints

$$F_j(\varphi_\sigma) = \gamma_j \quad (j = 1, \dots, s), \quad (2)$$

where  $\gamma_j$  are prescribed real numbers (we shall consider only admissible constraints, i.e., those compatible and not contradicting the membership of the functions  $\varphi_\sigma(x)$  in the class  $W_\sigma^{(1)}$ ). Then the following holds.

**Theorem.** There exists an extremal function satisfying the constraints (2), for  $s \leq 2$ , of the form

$$\varphi_\sigma^*(x) = [\psi_{\sigma/2}(x)]^2, \quad (3)$$

where  $\psi_{\sigma/2}(x)$  is some real entire function of degree  $\sigma/2$ .

**Remark.** 1) The uniqueness of the extremal function  $\varphi_\sigma^*(x)$  is not asserted here; 2) if  $s > 2$ , i.e., if more than two constraints of type (2) are prescribed, then the theorem may fail to hold.

By virtue of conditions (1) and (3), the function  $\psi_{\sigma/2}(x) \in W_\sigma^{(2)}$ , and therefore, by the Wiener–Paley theorem <sup>(1)</sup>, it can be represented in the form

$$\psi_{\sigma/2}(x) = \int_{-\sigma/2}^{\sigma/2} f(t)e^{ixt} dt, \quad (4)$$

where  $f(t) \in L_2[-\sigma/2, \sigma/2]$ , and

$$A_{\varphi_\sigma} \equiv \|\varphi_\sigma\| = \int_{-\infty}^{\infty} |\psi_{\sigma/2}(x)|^2 dx = \pi \int_{-\sigma/2}^{\sigma/2} |f(t)|^2 dt. \quad (5)$$

Thus, the problem of finding an extremal function  $\varphi_\sigma^*(x) \in W_\sigma^{(1)}$  subject to the constraints (2), and its norm, reduces to the minimization of the integral

$$\int_{-\sigma/2}^{\sigma/2} f^2(x) dx$$

under certain constraints imposed on the function  $f(x)$ , which follow from conditions (2) and relations (3) and (4).

To solve this problem, let us write for the function  $f(x)$  the Fourier series in the system  $\{\hat{P}_n(x)\}_0^\infty$  of Legendre polynomials normalized on  $[-\sigma/2, \sigma/2]$ :

$$f(x) \sim \sum_{k=0}^{\infty} a_k \hat{P}_k(x). \quad (6)$$

Taking into account that the system  $\{\hat{P}_n(x)\}_0^\infty$  is closed and that  $f(x) \in L_2[-\sigma/2, \sigma/2]$ , we obtain:

$$A_{\varphi_\sigma} = \pi \sum_{k=0}^{\infty} a_k^2, \quad (7)$$

and conditions (2) will be rewritten in the corresponding way in terms of the coefficients  $a_k$ .

Consider the following examples:

1. Let  $F(\varphi_\sigma) \equiv c_0 = 1$  ( $s = 1$ ). Then, by virtue of (3), (4), and (6),

$$\psi_{\sigma/2}(0) = \int_{-\sigma/2}^{\sigma/2} \sum_{k=0}^{\infty} a_k \hat{P}_k(x) dx = 1,$$

and, consequently,

$$a_0 = \sigma^{-1/2}. \quad (8)$$

Minimizing  $A_{\varphi_\sigma}$  (expression (7)) under condition (8), we find

$$a_k = 0 \quad (k = 1, 2, \dots); \quad f(x) = \frac{1}{\sigma}; \quad \varphi_\sigma^*(x) = \left( \frac{\sin \sigma x / 2}{\sigma x / 2} \right)^2; \quad A_{\varphi_\sigma^*} = \frac{\pi}{\sigma}.$$

The analogous problem for functions  $\varphi_\sigma(x) \in B_\sigma$  that are monotone on the entire real axis was solved by S. N. Bernstein (2).

2. Let  $F_1(\varphi_\sigma) \equiv c_0 = 1$  and  $F_2(\varphi_\sigma) \equiv c_2 = \gamma$  ( $s = 2$ ). Performing calculations similar to the preceding ones, we find

$$a_0 = \sigma^{-1/2}; \quad a_2 = -\frac{6\sqrt{5}}{\sigma^{5/2}} \left( \gamma + \frac{\sigma^2}{12} \right); \quad a_1 = 0; \quad a_k = 0 \quad (k = 3, 4, \dots);$$

$$f(x) = \frac{1}{\sigma} - \frac{6\sqrt{5}}{\sigma^{5/2}} \left( \gamma + \frac{\sigma^2}{12} \right) \hat{P}_2(x); \quad A_{\varphi_\sigma^*} = \pi \left[ \frac{1}{\sigma} + \frac{180}{\sigma^5} \left( \gamma + \frac{\sigma^2}{12} \right)^2 \right]$$

(the expression for  $\varphi_\sigma^*(x)$  is not given here because of its cumbersomeness, but it is easy to obtain it by taking (4) into account). For functions  $\varphi_\sigma \in B_\sigma$  that are monotone on the real axis, the analogous problem was solved by us earlier (3).

It is easy to see that, putting in this example  $c_2 = -\sigma^2/12$ , we obtain for  $A_{\varphi_\sigma^*}$  the minimal value coinciding with  $A_{\varphi_\sigma}$  in the first example (i.e., corresponding to the prescription of only one coefficient  $c_0 = 1$ ).

For real polynomials of finite degree  $\varphi_\sigma(x) \in W_\sigma^{(p)}$ , nonnegative on the real axis, the analogous theorem indicated here is also valid.

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Received  
22 X 1964

## REFERENCES

- <sup>1</sup> R. Paley, N. Wiener, *Fourier Transforms in the Complex Domain*, N. Y., 1934.
- <sup>2</sup> S. N. Bernstein, *Collected Works*, 2, No. 85, 1954.
- <sup>3</sup> B. A. Rymarenko, DAN, 155, No. 1 (1964).

*Note: Figure translations are in progress. See original paper for figures.*

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