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Abstract

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A LIMIT CAUCHY PROBLEM FOR DIFFERENTIAL EQUATIONS OF THE FIRST AND SECOND ORDERS WITH UNBOUNDED NONLINEAR POTENTIAL OPERATORS

(Presented by Academician I. N. Vekua on 27 III 1965)

In notes ([1,2]) we investigated solutions of the Cauchy problem for equations of parabolic and hyperbolic types. In the present work analogous questions are studied for a limit Cauchy problem for similar equations. The results are formulated in terms pertaining to general equations. Applications to differential equations (ordinary or with partial derivatives) can be obtained by standard schemes (see, for example, ([1,2])).

We shall say that an unbounded operator $A(t)$ $(0 < t < \infty)$, acting in a Hilbert space H and having a domain of definition $\mathfrak{D}$ independent of t, satisfies condition (A_0) if it is self-adjoint, positive definite, $A'(t)$ exists, and

$$(A'(t)x, x) \geq -a_0(t)(A(t)x, x) \quad (x \in \mathfrak{D}),$$

where $a_0(t)$ is some function on $(0, \infty)$.

Let $F(t, x)$ $(0 < t \leq \infty,\ x \in \mathfrak{D}(A^{1/2}))$ be a nonlinear functional differentiable in the sense of Gateaux. We denote its gradient by $P(t, x)$.

1. We first consider the limit problem for the first-order differential equation

$$\dot{x}(t) = A(t)x(t) + P[t, x(t)], \tag{1}$$

$$\lim_{\tau\to\infty} x(\tau) = x(\infty) = x_\infty \tag{2}$$

in the Hilbert space H.

Theorem 1. *Let the operator $A(t)$ satisfy condition (A_0). Suppose that $\partial F(t,x)/\partial t$ exists and that the condition*

$$\partial F(t,x)/\partial t \geq -a_0(t)F(t,x) \quad (0 < t < \infty,\ x \in \mathfrak{D}(A^{1/2})) \tag{3}$$

is fulfilled. Then, for any solution $x(t)$ of problem (1), (2), for $t > 0$ the estimate

$$\|A^{1/2}(t)x(t)\| + 2F[t, x(t)] \leq$$

$$\leq [\|A^{1/2}(\infty)x_\infty\|^2 + 2F(\infty, x_\infty)] \exp\left[\int_t^\infty a_0(s)\,ds\right]$$

holds.

From this theorem, as a consequence, one can obtain a number of facts concerning boundedness, Lyapunov stability of solutions of problem (1), (2), and their behavior as $t \to 0$. Here the role of a Lyapunov function is played by the functional

$$v(t,x) = \|A^{1/2}(t)x\|^2 + 2F(t,x).$$

Theorem 2. Let the operator $A(t)$ satisfy condition (A_0). Let the functional $F(t,x)$ satisfy condition (3), and let its gradient $P(t,x)$ satisfy the condition

$$\|P(t,x) - P(t,y)\| \leq K_r(t)\|A^{1/2}(t)(x-y)\|$$

$$(\|A^{1/2}x\|,\ \|A^{1/2}y\| \leq r, \quad 0 < t < \infty), \tag{4}$$

where $K_r(t)$ is a square-summable function. Finally, let $x(t)$ be a solution of problem (1), (2); $y(t)$ a solution of equation (1) satisfying the condition $y(\infty) = y_\infty$.

Then, for $t > 0$, the estimate holds

$$\|A^{1/2}(t)[x(t) - y(t)]\| \leq C\|A^{1/2}(\infty)(x_\infty - y_\infty)\| \exp\left[2\int_t^\infty \alpha_0(s)\,ds\right].$$

From this estimate follow the uniqueness and stability of the solutions of problem (1), (2).

By $y_0(t)$ we denote a differentiable solution of the equation

$$A(t)y(t) + P[t, y(t)] = 0. \tag{5}$$

We shall say that the solutions of problem (1), (2) are **stabilizable** (cf. (3)) to the solution of equation (5), if

$$\lim_{t\to 0} \|A^{1/2}(t)[x(t) - y_0(t)]\| = 0.$$

Theorem 3. Let $A(t)$ and $F(t,x)$ satisfy the conditions of Theorem 2. Let $\lim\limits_{t\to 0} \varepsilon_0(t) = 0$, where

$$\varepsilon_0(t) = \|A^{1/2}(\infty)(x_\infty - y_0(\infty))\|^2 \exp\left\{\int_t^\infty [\alpha_0(s) + K_r^2(s)]\,ds\right\} - \int_t^\infty \|\dot{y}_0(s)\|^2 \exp\left\{\int_s^\infty [\alpha_0(\tau) + K_r^2(\tau)]\,d\tau\right\}\,ds$$

Then the solutions of problem (1), (2) are stabilizable to the solution of equation (5).

Let us consider the question of the existence of a solution of problem (1), (2). Let the differentiable functions $x_n(t)$ from $\mathfrak{D}(A)$ be defined by the equalities

$$x_n(t) = x_\infty \qquad \left(n - \frac{1}{n} \le t \le \infty\right),$$

$$\dot{x}_n(t) = A(t)x_n(t) + P\left[t, x_n\left(t + \frac{1}{n}\right)\right] \qquad \left(0 < t < n - \frac{1}{n}\right). \tag{6}$$

If the sequence of functions $x_n(t)$, defined by the equalities (6), converges uniformly in t (in the norm $\|A^{1/2}(t)x\|$) to some function $x^*(t)$, then this function will be called a **generalized solution** of problem (1), (2).

Theorem 4. Let $A(t)$ and $F(t,x)$ satisfy the conditions of Theorem 2. In addition, let $P(t,x)$ be a uniformly continuous operator (in the norm $\|A^{1/2}(t)x\|$).

Then, if the sequence of functions defined by the equalities (6) is equicontinuous, there exists a generalized solution of problem (1), (2), and if a classical solution of this problem exists, then it coincides with the generalized solution.

2. We now consider the limiting Cauchy problem for an equation of second order:

$$\ddot{x}(t) + A(t)x(t) + P[t, x(t)] = 0, \tag{7}$$

$$x(\infty) = x_\infty, \qquad \dot{x}(\infty) = \dot{x}_\infty,$$

$$\ddot{y}(t) + A(t)y(t) + P[t, y(t)] = B[t, \dot{y}(t)], \tag{8}$$

$$y(\infty) = y_\infty, \qquad \dot y(\infty) = \dot y_\infty$$

also in the Hilbert space H.

Theorem 5. *Let the operator $A(t)$ satisfy condition (A_0), and let the functional $F(t,x)$ satisfy condition (3). Let the operator $B(t,\dot y)$ satisfy the condition*

$$(B(t)\dot y, \dot y) \geqslant -\frac{\alpha_0(t)}{2}\|\dot y\|^2 \quad (x \in \mathfrak{D}(B)).$$

Then, for any solution $x(t)$ of problem (7) and $y(t)$ of problem (8), for $t>0$ the estimates hold

$$\|\dot x(t)\|^2 + \|A^{1/2}(t)x(t)\|^2 + 2F[t,x(t)] \leqslant$$

$$\leqslant [\|\dot x_\infty\|^2 + \|A^{1/2}(\infty)x_\infty\|^2 + 2F(\infty,x_\infty)]\exp\left[\int_t^\infty |\alpha_0(s)|\,ds\right];$$

$$\|\dot y(t)\|^2 + \|A^{1/2}(t)y(t)\|^2 + 2F[t,y(t)] \leqslant$$

$$\leqslant [\|\dot y_\infty\|^2 + \|A^{1/2}(\infty)y_\infty\|^2 + 2F(\infty,y_\infty)]\exp\left[\int_t^\infty \alpha_0(s)\,ds\right].$$

From these estimates one can obtain a number of facts concerning boundedness and Lyapunov stability of solutions of problems (7), (8), and also concerning the behavior of solutions of problem (8) as $t \to 0$. Here the role of the Lyapunov function is played by the functional

$$v(t,x) = \|\dot x\|^2 + \|A^{1/2}(t)x\|^2 + 2F(t,x).$$

Theorem 6. *Let the operator $A(t)$ satisfy condition (A_0). Let the functional $F(t,x)$ satisfy condition (3), and its gradient $P(t,x)$, condition (4). Finally, let $x_1(t)$ be a solution of problem (7), and let $x_2(t)$ be a solution of the problem*

$$\ddot x + A(t)x + P(t,x) = 0, \qquad x(\infty) = y_\infty, \qquad \dot x(\infty) = \dot y_\infty.$$

Then, for $t>0$, the estimate holds

$$\|\dot x_1(t) - \dot x_2(t)\|^2 + \|A^{1/2}(t)[x_1(t)-x_2(t)]\|^2 \leqslant$$

$$\leqslant [\|\dot{x}_\infty - \dot{y}_\infty\|^2 + \|A^{1/2}(\infty)(x_\infty - y_\infty)\|^2] \exp\left\{\int_t^\infty [|\alpha_0(s)| + K_r(s)]\, ds\right\}.$$

It is easy to see that consequences of this theorem may be theorems of uniqueness and stability of the solution of problem (7).

Let $y_0(t)$ be a twice differentiable solution of the equation

$$A(t)y_0(t) + P[t, y_0(t)] = B[t, \dot{y}_0(t)]. \tag{9}$$

We shall say that the solutions $y(t)$ of problem (8) are stabilizable to the solution $y_0(t)$ of equation (9) if

$$\lim_{t\to 0}\{\|\dot{y}(t) - \dot{y}_0(t)\|^2 + \|A^{1/2}(t)[y(t) - y_0(t)]\|^2\} = 0.$$

Theorem 7. Let $A(t)$ and $F(t, x)$ satisfy the conditions of Theorem 6. Let $B(t, \dot{y})$ satisfy the condition

$$(B(t, \dot{y}_1) - B(t, \dot{y}_2),\ \dot{y}_1 - \dot{y}_2) \geq \frac{1 - \alpha_0(t)}{2}\|\dot{y}_1 - \dot{y}_2\|^2.$$

Finally, suppose that $\lim_{t\to 0} \delta_0(t) = 0$, where

$$\delta_0(t) = \left\{\|\dot{y}_0(\infty) - \dot{y}_\infty\|^2 + \left\|A^{1/2}(\infty)\,[y_0(\infty) - y_\infty]\right\|^2\right\} \times$$

$$\times \exp\left\{\int_t^\infty [\alpha_0(s) + K_r(s)]\, ds\right\} - \int_t^\infty \|\ddot{y}_0(s)\|^2 \exp\left\{\int_s^t [\alpha_0(\tau) + K_r(\tau)]\, d\tau\right\} ds.$$

Then the solutions of problem (8) are stabilizable to a solution of equation (9).

In conclusion we consider the question of the existence of a solution of equation (9).

We shall assume that a sequence of twice differentiable solutions $x_n(t)$ $(n = 1, 2, ...)$ from $\mathfrak{D}(A)$ is defined by the equalities

$$x_n(t) = x_\infty \quad \left(n - \frac{1}{n} \leq t \leq \infty\right), \qquad \dot{x}_n(\infty) = \dot{x}_\infty,$$

$$\ddot{x}_n(t) + A(t)x_n(t) + P\left[t, x_n\left(t + \frac{1}{n}\right)\right] = 0 \quad \left(0 < t < n - \frac{1}{n}\right). \tag{10}$$

If the sequence of functions $x_n(t)$ defined by the equalities (10) converges uniformly in t (in the norm $\|\dot{x}\| + \|A^{1/2}(t)x\|$) to some function $x^*(t)$, then we shall call this function a **generalized solution** of problem (7).

Theorem 8. Let $A(t)$ and $F(t, x)$ satisfy the conditions of Theorem 6. Let, moreover, $P(t, x)$ be a uniformly continuous operator (in the norm $\|\dot{x}\| + \|A^{1/2}(t)x\|$).

Then, if the sequence of functions $x_n(t)$ defined by the equalities (10) is equicontinuous, there exists a generalized solution of problem (7), and if a classical solution of this problem exists, then it coincides with the generalized solution.

It should be noted that the behavior of the solution of problem (7) at infinity, in another aspect, in connection with the investigation of the scattering operator, was studied in the work [4].

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Note: Figure translations are in progress. See original paper for figures.