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## Abstract

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## GEOPHYSICS

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# ON DETERMINING THE BOUNDARIES OF CLOUDINESS

In the hydrodynamic theory of cloudiness arising on the leeward side of an obstacle, the boundary of the cloud is usually determined, after the adiabatic motion around the obstacle has been found, as the locus of points at which the moisture transported by the adiabatic motion begins to saturate the space ($^1$). Meanwhile, inside the cloud, if precipitation does not fall out, the motion should be regarded as pseudoadiabatic: instead of conservation of potential temperature we must assume conservation of pseudopotential temperature. The system of equations of hydrodynamics is then modified, and the entire structure of the solution may be disturbed. The correct formulation of the problem will be the consideration of an adiabatic flow in which separate regions with a pseudoadiabatic process are embedded; the boundaries of these regions are not known in advance and must be determined in parallel with the solution of the problem of the motion itself.

As a boundary condition on the boundary of these regions one may adopt the saturation condition. Introducing the specific humidity $q$, we shall write on the boundary $q = q_{\max}(T, p)$, where $q_{\max}$ is determined by the well-known Magnus formula and depends on the temperature $T$ and pressure $p$ ($^2$):

$$q_{\max} = 0.23 \cdot 10^{-3} \frac{P}{p} e_{\max}(T); \qquad e_{\max}(T) = 6.1 \cdot 10^{-3} P \exp 17.13 \frac{T - 273}{T - 38}. \quad (1)$$

For simplicity let us consider the case of steady flow around a ridge. The motion takes place in the plane $(x, z)$ ($x$ is the horizontal coordinate, $z$ the vertical coordinate). One may introduce the stream function $\psi$ from the relations

$$u = \frac{\partial \psi}{\partial z}, \qquad w = -\frac{\partial \psi}{\partial x}. \quad (2)$$

We do not take into account the effect of air compressibility and confine the motion below by the contour being flowed around, and above by the horizontal wall $z = H$. Let, in addition, the value $u_\infty$ of the horizontal velocity $u$ at $x = -\infty$ and the value $T_\infty$ of the temperature $T$ be given: $u_\infty = U(z)$, $T_\infty = T_0 - \gamma z$

($\gamma, T_0$ are constants). The equations of motion, after eliminating the pressure from them, give the relation ($^3$)

$$\frac{\partial\psi}{\partial z}\frac{\partial\Omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\Omega}{\partial z}=\frac{g}{T_m}\frac{\partial T'}{\partial x}, \tag{3}$$

where $\Omega=\partial u/\partial z-\partial w/\partial x=\Delta\psi$, $T'$ is the deviation of the temperature from $T_\infty$; $g$ is the acceleration of gravity; $T_m$ is the mean air temperature ($T_m\approx 250°$). We shall write the heat-inflow equation in the form:

$$\frac{\partial\psi}{\partial z}\frac{\partial T'}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T'}{\partial z}-(\gamma_a\varepsilon-\gamma)\frac{\partial\psi}{\partial x}=0, \tag{4}$$

where $\gamma_a=[(\varkappa-1)/\varkappa]g/R$ ($R$ is the gas constant, $\varkappa$ is the ratio of heat capacities); here $\varepsilon=1$ outside the cloud and (see ($^2$))

$$\varepsilon=\left[1+\frac{0.623}{c_p}L\frac{\varkappa}{\varkappa-1}\frac{e_{\max}(T)}{pT}\right]\left(1+\frac{0.623}{c_p}L\frac{1}{p}\frac{de_{\max}}{dT}\right)^{-1}. \tag{5}$$

inside the cloud ($L$ is the latent heat of condensation, $c_p$ is the heat capacity of air at constant pressure).

To determine the humidity $q$, we adopt the transport equation

$$\frac{\partial\psi}{\partial x}\frac{\partial q}{\partial z}-\frac{\partial\psi}{\partial z}\frac{\partial q}{\partial x}=0, \tag{6}$$

which is integrated and gives $q=Q(\psi)$, where $Q$ is an arbitrary function of $\psi$, whose form is found from the condition $q=q_\infty(z)$ at infinity ahead of the obstacle.

Substituting $Q(\psi)$ into the left-hand side of (1), we obtain, at the cloud boundary, a relation between $\psi$, $p$, and $T$. With a high degree of accuracy one may here replace $p$ in (1) by its value

$$p_\infty=p_0\exp\left(\frac{g}{R}\int_z^0\frac{dz}{T_\infty}\right)$$

($p_0$ is the pressure at sea level) and write, instead of (1),

$$Q(\psi)=3.8\cdot 10^{-3}\frac{P}{p_0}\exp\left(\int_0^{z_{\text{cr}}}\frac{g}{R}\frac{dz}{T_\infty}+17.13\,\frac{T_\infty-273+T'_{\text{cr}}}{T_\infty-38+T'_{\text{cr}}}\right), \tag{7}$$

($\psi_{\text{cr}}$, $z_{\text{cr}}$, $T'_{\text{cr}}$ are, respectively, $\psi$, $z$, $T'$ at the cloud boundary). Equation (7) relates $T'_{\text{cr}}$, $z_{\text{cr}}$, and $\psi_{\text{cr}}$.

**Fig. 1.**

However, the temperature $T'$ can be eliminated by means of (4). Namely, for the region outside the cloud, (4) admits, as is known, the integral

$$T' = -(\gamma_a - \gamma)\left[z - f_1(\psi)\right], \tag{8}$$

where $f_1$ is an arbitrary function of $\psi$, whose form is found from the conditions at infinity. On the other hand, with the same accuracy with which (5) was written, we may replace in $\varepsilon$ the functions $p$ and $T$ by their values $p_\infty$ and $T_\infty$.* Then we can write for the cloud

$$T' = -(\gamma_a - \gamma)\left[\int_0^z \frac{\varepsilon_\infty \gamma_a - \gamma}{\gamma_a - \gamma}\, dz - F_1(\psi)\right], \tag{9}$$

where $F_1(\psi)$ is another arbitrary function of $\psi$.

Substituting $T'$ from (8) into (3), we obtain, outside the cloud, one equation for $\psi$, admitting an integral of the form

$$\Delta\psi = \left[f_2(\psi) + z\frac{df_1}{d\psi}\right]\frac{g}{T_m}(\gamma_a - \gamma), \tag{10}$$

where $f_2(\psi)$ is a new arbitrary function of $\psi$.

Inside the cloud

$$\Delta\psi = \left[F_2(\psi) + z\frac{dF_1}{d\psi}\right]\frac{g}{T_m}(\gamma_a - \gamma), \tag{11}$$

where $F_2(\psi)$ is an arbitrary function of $\psi$.

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* In practice, replacing $p_\infty$ by the approximate formula

$$p_\infty = p_0 \exp\left(-\frac{g}{RT_m}\right)$$

we can obtain for $\varepsilon$ the approximate expression

$$\varepsilon \approx \varepsilon_\infty = (1 + 0.133\exp\chi)(1 + 0.851\exp\chi)^{-1},$$

where

$$\chi - \frac{20.2}{T_m}\left[T_0 - 273 - \left(\gamma - \frac{g}{20.2\,R}\right) z\right].$$

Condition (7) will be used below in the approximate form*

$$Q(\psi_{\text{cr}}) = 3.8 \cdot 10^{-3} \frac{P}{p_0} \exp(bt_0 + bT'_{\text{cr}} - \alpha z_{\text{cr}}), \tag{12}$$

where

$$\alpha = b\gamma - \frac{g}{RT_m}, \qquad b = \frac{20.2}{T_m}, \qquad t_0 = T_0 - 273.$$

If $q_\infty$ is represented in the form $q_\infty = q_0 \exp(-\Gamma z)$, where $q_0$ and $\Gamma$ are constants, then, by (8), since $T'$ tends to zero far from the obstacle and we have $[f_1(\psi)]_{x\to-\infty} = z$, one may write $q_\infty = q_0 \exp(-\Gamma f_1(\psi_\infty))$. Then $Q(\psi) = q_0 \exp(-\Gamma f_1(\psi))$. Substituting this $Q$ into (12), replacing $T'$ according to (8), we obtain the final condition relating $z_{\text{cr}}$ and $\psi_{\text{cr}}$ in the form

$$z_{\text{cr}} = m f_1(\psi_{\text{cr}}) + s, \tag{13}$$

where

$$m = [\Gamma + (\gamma_a - \gamma) b]\left(b\gamma_a - \frac{g}{RT_m}\right)^{-1}, \qquad s = \left(bt_0 + \ln\frac{0.38P}{q_0 p_0}\right)\left(b\gamma_a - \frac{g}{RT_m}\right)^{-1}. \tag{14}$$

Next, letting $x \to -\infty$, from (10) we have

$$f_2(\psi) = -f_1 \frac{df_1}{d\psi} + \frac{T_m}{g(\gamma_a - \gamma)} \left(\frac{dU}{dz}\right)_{z = f_1(\psi)}. \tag{15}$$

At the boundary of the cloud we shall require continuity of the transition of the functions $\psi$ and $T'$, and also of the velocity and the vorticity $\Omega$. These matching conditions will first of all make it possible to relate $F_1, F_2$ to $f_1$. Namely, equating $T'$ according to (8) and (9) under (13), we obtain:

$$F_1(\psi) = \int_0^{m f_1(\psi) + s} \frac{\varepsilon_\infty \gamma_a - \gamma}{\gamma_a - \gamma}\, dz + (1 - m) f_1(\psi) - s. \tag{16}$$

From the equality of the vorticities (10) and (11) at the matching, we obtain

$$F_2(\psi) = [mf_1(\psi) + s]\left(\frac{df_1}{d\psi} - \frac{dF_1}{d\psi}\right) + f_2(\psi). \tag{17}$$

Thus $F_1, F_2, f_2$ are expressed through $f_1$. To determine the latter, according to (2), the equation is:

$$\psi = \int_0^{f_1(\psi)} U(z)\, dz. \tag{18}$$

Finally, introducing along the way dimensionless quantities, we have, outside the cloud,

$$\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial Z^2} = D^2(Z - f) + \left(\frac{d\overline{U}}{dZ}\right)_{Z=f}; \tag{19}$$

inside the cloud

$$\frac{\partial^2\Psi}{\partial X^2} + \frac{\partial^2\Psi}{\partial Z^2} = D^2[(1 - mn)Z + Smn - (1 - m^2 n)f] + \left(\frac{d\overline{U}}{dZ}\right)_{Z=f}, \tag{20}$$

where

$$n = \frac{\gamma_a}{\gamma_a - \gamma}\,[1 - \varepsilon_\infty(Z)]_{Z=mf(\Psi)+S},$$

$$D^2 = g\,\frac{\gamma_a - \gamma}{T_m}\frac{H^2}{V^2}, \qquad HZ = z, \qquad HX = x, \qquad HS = s, \qquad Hf = f_1 \tag{21}$$

$$VH\Psi = \psi, \qquad V\overline{U} = U$$

($V$ is the characteristic velocity).

* We approximately replace the integration and discard $T'$ in the denominator of the right-hand side of (7).

As an example, let us consider the case of a very gently sloping obstacle, when the long-wave method can be applied and, in equations (19), (20), the second derivatives with respect to $X$ can be neglected. We shall also assume that $U(z) = \text{const} = V$. Then $f(\Psi) = \Psi$. For $n$ we shall take a constant (mean) value. In the absence of cloudiness we obtain here simply

$$\Psi = Z - Z_0 \csc D(1 - Z_0) \sin D(1 - Z) \tag{22}$$

($Z = Z_0(X)$ is the equation of the contour being flowed around).

If, however, a cloud forms over the obstacle, we must set: below the cloud

$$\Psi = Z - \csc D\,(Z_1 - Z_0) \left[ Z_0 \sin D(Z_1 - Z) + \left( \frac{1-m}{m} Z_1 - \frac{S}{m} \right) \sin D(Z_0 - Z) \right]; \tag{23}$$

in the cloud

$$\begin{aligned} (1 - m^2 n)\Psi &= mnS + (1 - mn)Z + \\ &+ \csc \widetilde{D}\,(Z_2 - Z_1) \left[ \left( \frac{1-m}{m} Z_1 - \frac{S}{m} \right) \sin \widetilde{D}(Z_2 - Z) + \right. \\ &\left. + \left( \frac{1-m}{m} Z_2 - \frac{S}{m} \right) \sin \widetilde{D}(Z - Z_1) \right]; \end{aligned} \tag{24}$$

above the cloud

$$\Psi = Z + \left( \frac{1-m}{m} Z_2 - S \right) \csc D(1 - Z_2) \sin D(1 - Z). \tag{25}$$

Here $Z_1$ and $Z_2$ are, respectively, the lower and upper boundaries of the cloud. $\widetilde{D} = \sqrt{1 - m^2 n}\, D$. The quantities $Z_1$ and $Z_2$ are determined from a system of two transcendental equations (we do not write them out), which is obtained if: a) $\partial\Psi/\partial Z$ from (23) is equated to $\partial\Psi/\partial Z$ from (24) at $Z = Z_1$, and b) $\partial\Psi/\partial Z$ from (24) is equated to (25) at $Z = Z_2$.

The solution will have a sharply different character depending on whether $n > 1$ ($\varepsilon\gamma_a < \gamma$) or $n < 1$ ($\varepsilon\gamma_a > \gamma$). Figure 1 gives an example of the cloudiness contour when flowing around the obstacle $Z_0 = 0.42e^{-X}$; it was assumed that $D = 3$, $m = 0.6$, $S = 0.432$. The dotted line gives the cloudiness contour for the case of everywhere adiabatic motion; the two other lines depict cloudiness for $n = 0.9$ and $n = 1.5$, respectively.

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*Note: Figure translations are in progress. See original paper for figures.*