

# GENERALIZED EXTENSIONS OF UNBOUNDED OPERATORS

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## Abstract

## Full Text

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MATHEMATICS

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# GENERALIZED EXTENSIONS OF UNBOUNDED OPERATORS

(Presented by Academician A. N. Kolmogorov on 27 III 1965)

**I.** Let  $H_0$  be a complete Hilbert space with scalar product  $(f, g)_0$  and norm  $\|f\|_0$ . Suppose that in  $H_0$  there is an everywhere dense linear set  $H_+$ , which is a complete Hilbert space with respect to another scalar product  $(f, g)_+$  and norm  $\|f\|_+$ . We shall assume that

$$\|f\|_0 \leq \|f\|_+ \quad (f \in H_+).$$

We shall call the space  $H_+$  a space with positive\* norm, and also a space of basic elements. We shall say that every antilinear functional  $a(f)$  on  $H_+$  is generated by a generalized element  $a$ , and we shall write the value of the functional on the element  $f \in H_+$  as  $a(f) = (a, f)_0$ . In what follows we use the notation  $\overline{(a, f)_0} = (f, a)_0$ . Obviously, the totality of all generalized elements is a linear set. Since the space  $H_+$  is Hilbert, every antilinear functional on  $H_+$  can be represented in the form of a scalar product defined in  $H_+$ , i.e.,

$$(a, f)_0 = (a^*, f)_+ = (y_a, f)_+ \quad (a^* = y_a, a^* \in H_+, f \in H_+). \quad (1)$$

Equality (1) generates a linear operator  $J$ , mapping the set of generalized elements into the space  $H_+$ . The range of the operator is, obviously, all of  $H_+$ .

We introduce in the set of generalized elements  $H_-$  a scalar product, putting

$$(a, \beta)_- = (y_a, y_\beta)_+ \quad (a, \beta \in H_-). \quad (2)$$

The set  $H_-$  with the scalar product (2) is a Hilbert space.

It follows from (2) that the operator  $J$  is isometric, mapping  $H_-$  into  $H_+$ . Since

$$\|a\|_- = \sup_{f \in H_+} \frac{|(a, f)_0|}{\|f\|_+},$$

the inequality holds

$$\|f\|_- \leq \|f\|_0 \leq \|f\|_+.$$

Thus,

$$H_+ \subseteq H_0 \subseteq H_-.$$

We shall call the space  $H_-$  a space with negative norm, and also the space of generalized elements of the Hilbert space  $H_0$ . It is not difficult to prove that  $H_+$  is dense in  $H_-$ .

II. Let  $\Omega(f, g)$  be a bilinear functional defined in  $H_+$ . Then it is easy to show that

$$\Omega(f, g) = (Bf, g)_0 \quad (f, g \in H_+), \quad (3)$$

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\* In the exposition of Sec. I we mainly follow the paper of Yu. M. Berezanskii (2).

where  $B$  is a bounded linear operator acting from  $H_+$  into  $H_-$ , which is uniquely determined by the bilinear functional  $\Omega$ .

Let  $B$  be an arbitrary bounded linear operator acting from  $H_+$  into  $H_-$ . The expression  $(f, Bg)_0$  ( $f, g \in H_+$ ) is, obviously, a bilinear functional in  $H_+$ . Then, according to (3), there is a uniquely determined bounded linear operator  $B^\times$ , mapping  $H_+$  into  $H_-$ , for which

$$(f, Bg)_0 = (B^\times f, g)_0 \quad (f, g \in H_+).$$

We shall call the operator  $B^\times$  the **generalized adjoint operator** with respect to  $B$ . If  $B = B^\times$ , then such an operator will be called a **generalized self-adjoint operator**.

III. Let  $T$  be a closed operator acting in the Hilbert space  $H_0$ , for which

$$T_0 \subset T, \quad T^* \subset T_0^*,$$

where  $T_0$  is a symmetric operator with dense domain of definition.\* Consider the Hilbert space  $H_+ = D_{T_0^*}$  with scalar product

$$(f, g)_+ = (T_0^* f, T_0^* g)_0 + (f, g)_0 \quad (f, g \in H_+).$$

Construct the triple of spaces  $H_+ \subseteq H_0 \subseteq H_-$ . We now pose the following problem: can the operators  $T$  and  $T^*$  be extended linearly to all of  $H_+ = D_{T_0^*}$  so that the obtained extensions  $T_{H_+}$  and  $T_{H_+}^\times$  are adjoint to one another in the generalized sense? It turns out that this can always be done; moreover, the obtained generalized extensions  $T_{H_+}$  and  $T_{H_+}^\times$  will be linear bounded operators acting from  $H_+$  into  $H_-$ , and, in contrast to  $T$  and  $T^*$ , will have the same domain of definition  $H_+$ . In the paper a number of properties of generalized extensions are established, their resolvent is investigated, and an estimate for its norm in the space  $H_-$  is found.

IV. **Theorem 1.** \*If the operator  $T$  belongs to the class  $\Omega$ , then  $T$  and  $T^*$  can always be extended\*\* to all of  $H_+ = D_{T_0^*}$  so that the obtained extensions  $T_{H_+}(H_+ \rightarrow H_-)$  and  $T_{H_+}^\times(H_+ \rightarrow H_-)$  are adjoint to one another in the generalized sense.\*

**Definition.** A linear extension  $T_{H_+}$  of the operator  $T$  to  $H_+ = D_{T_0^*}$  will be called **generalized** if  $T_{H_+}^\times$  is an extension of  $T^*$  to  $H_+$ . It follows from Theorem 1 that if the operator  $T$  belongs to the class  $\Omega$ , then it has a nontrivial generalized extension  $T_{H_+}$ , which, as examples show, is not determined uniquely by the operator. However, the following holds.

**Theorem 2.** Let  $T_{H_+}$  and  $T'_{H_+}$  be generalized extensions of the operator  $T$  to the space  $H_+ = D_{T_0^*}$ , and let

$$T_{H_+} = A + iB; \quad T'_{H_+} = A' + iB',$$

where  $A, B, A', B'$  are linear generalized self-adjoint operators acting from  $H_+$  into  $H_-$ . Then, if  $B = B'$ , then  $A = A'$ .

We shall regard the generalized extension of the operator  $T$  as an operator acting in the Hilbert space  $H_-$ . This operator has in  $H_-$  the dense domain of definition  $H_+$ .

**Theorem 3.** If  $\lambda$  is a regular point of an operator  $T$  belonging to the class  $\Omega$ , then it is a point of regular type for any generalized extension  $T_{H_+}(H_+ \rightarrow H_-)$  of the operator  $T$ .

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\* An operator  $T$  possessing the indicated properties will henceforth be assigned to the class  $\Omega$ .

\*\* In the theorem the existence is proved of a nontrivial generalized extension, different from that which is obtained if  $T$  and  $T^*$  are extended by zero.

**Corollary 1.** If  $\tilde{T}_{H_+}$  is an arbitrary generalized extension of the operator  $T$ , then at every point  $\lambda$  regular for  $T$  there exists the resolvent  $(\tilde{T}_{H_+} - \lambda I)^{-1}$ , whose closure is a bounded operator defined everywhere in  $H$ , and for which

$$\|(\overline{T_{H_+} - \lambda I})^{-1}\| \leq (1 + |\lambda|) (1 + \|(T - \lambda I)^{-1}\|_0). \quad (4)$$

**Corollary 2.** If the number  $\lambda$  is not a point of regular type for some generalized extension  $T_{H_+}$ , then it belongs to the spectrum of the operator  $T$ .

**Corollary 3.** If the operator  $T$  has no spectrum in the finite part of the plane, then every number  $\lambda$  is a point of regular type for every generalized extension  $T_{H_+}$ , and, consequently, in this case there exists the resolvent  $(T_{H_+} - \lambda I)^{-1}$ , whose closure is a bounded operator in  $H$ , and for which (4) holds.

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*Note: Figure translations are in progress. See original paper for figures.*

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