ON THE ELEMENTARY THEORY OF CLASSICAL LIE ALGEBRAS

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Abstract

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ON THE ELEMENTARY THEORY OF CLASSICAL LIE ALGEBRAS

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1. Yu. L. Ershov has proved the hereditary undecidability of the elementary theories of various classes of matrix groups. In this note analogous results are obtained for classical Lie algebras.

Let L denote the set of all closed formulas of first-order predicate calculus with equality, containing no nonlogical constants other than the function symbols $+$ and $\times$.

Let $U(L)$ denote the collection of all formulas from L that are true under every interpretation; let $B(M)$ denote the collection of all such formulas from L that are true on every model from the class M of models of signature $\langle +, \times \rangle$. In what follows the symbol $+$ will be interpreted as the operation of addition of vectors, and the symbol $\times$ as the operation of vector multiplication.

In the present note it is shown that, whatever the class M of simple finite-dimensional complex Lie algebras containing Lie algebras of arbitrarily large finite dimensions, there is no recursive set of formulas Φ satisfying the condition

$$U(L) \subseteq \Phi \subseteq B(M).$$

In particular, the set $B(M)$ is nonrecursive, and the elementary theory of the class M is undecidable.

To each real finite-dimensional Lie algebra K we associate a complex algebra $[K]$ in the following way. Consider the totality of all expressions of the form $z = x + iy$, where $x \in K$, $y \in K$, $i = \sqrt{-1}$. Define on this totality the operations of addition, multiplication, and multiplication by complex numbers by the rules:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2);$$

$$(x_1 + iy_1) \times (x_2 + iy_2) = [(x_1 \times x_2) + (y_2 \times y_1)] + i[(x_1 \times y_2) + (y_1 \times x_2)],$$

$$(\beta + i\gamma)(x + iy) = (\beta x - \gamma y) + i(\gamma x + \beta y).$$

It is known that in this way one obtains a complex Lie algebra $[K]$; the dimensions of the algebras K and $[K]$ (the first over the field of real numbers, and the second over the field of complex numbers) coincide, and the algebras K and $[K]$ are simultaneously simple or nonsimple.

There exists a well-known effective procedure which to each formula $\mathfrak{A} \in L$ assigns a formula $\varphi\mathfrak{A} \in L$ such that $\varphi(\mathfrak{A})$ is true on K if and only if $\mathfrak{A}$ is true on $[K]$. From the effectiveness of the construction φ and the result mentioned at the beginning of this note, it follows that, whatever the class M of finite-dimensional real simple Lie algebras containing Lie algebras of arbitrarily large finite dimensions, there is no recursive set of formulas Φ satisfying the condition

$$U(L) \subseteq \Phi \subseteq B(M).$$

It follows from this that the elementary theory of every class of isotypic models containing the class M is undecidable. In particular, the elementary theories of the classes of compact Lie algebras, simple real Lie algebras, etc., are undecidable. From the latter it follows that the classes of compact Lie algebras, etc., and even the elementary theories of these classes are not finitely or recursively axiomatizable.

2. Adhering to the terminology adopted in (1), we shall set out some information on the structure of complex finite-dimensional simple Lie algebras.

Let $\widetilde{R}$ be a finite-dimensional complex simple Lie algebra. Then $\widetilde{R} = [R]$, where R is a compact simple real Lie algebra. Let S be a regular subalgebra of the algebra R, and let $\Sigma \subset S$ be the root system of the algebra R. There exists a $c \in S$ such that, for $x \in \widetilde{R}$, one has $x \in [S]$ if and only if $x \times c = 0$. If $\alpha \in \Sigma$, then $-\alpha \in \Sigma$. In Σ one can choose a subsystem Π consisting of simple root vectors. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$. Then $\{\alpha_1, \dots, \alpha_n\}$ is a basis of the linear space S over the field of real numbers. Every vector in Σ is an integral linear combination of vectors from Π, all coefficients in this combination being of the same sign. Setting $p_a(x) = a \times x$ for $a, x \in R$, $(x, y) = -\operatorname{Sp}(p_x p_y)$, we make the space R, together with the scalar product (x, y) thus introduced, Euclidean.

To each $\alpha \in \Sigma$ one can associate a vector $r_\alpha \in \widetilde{R}$ such that the collection of vectors $\{\alpha_1, \dots, \alpha_n; r_\alpha, \ \alpha \in \Sigma\}$ forms a basis of the linear space $\widetilde{R}$, and for $s \in [S]$, $\alpha, \beta \in \Sigma$ we have

$$s \times r_\alpha = i(\alpha, s) r_\alpha; \qquad r_\alpha \times r_{-\alpha} = i\alpha; \qquad r_\alpha \times r_\beta = N_{\alpha\beta} r_{\alpha+\beta},$$

if $\alpha + \beta \in \Sigma$, where $N_{\alpha\beta}$ is a complex number;

$$r_\alpha \times r_\beta = 0,$$

if $\alpha + \beta \notin \Sigma$.

If $y \in \widetilde{R}$ and for every $x \in [S]$ one has $x \times y = N_{xy}$, where N_x is a complex number, then either $y \in [S]$, or there exist $\alpha \in \Sigma$ and a complex number M such that $y = M r_\alpha$.

If $\alpha, \beta \in \Pi$, then there exists $\gamma \in \Sigma$ such that $(\alpha, \gamma) \neq 0$ and $(\beta, \gamma) \neq 0$ (see (^{2})).

3. Choose $h \in [S]$ such that $(\alpha_j, h) = -i$ for $j = 1, \dots, n$. An element $x \in \widetilde{R}$ satisfies the condition $h \times x = x$ if and only if there exist complex numbers $N_1, \dots, N_n$ such that $x = N_1 r_{\alpha_1} + \dots + N_n r_{\alpha_n}$.

Now consider the formula

$$\mathfrak{A}_1(x, c, h) \underset{df}{\Longleftrightarrow} \forall y\, (c \times y = 0 \to \forall z\, (([y \times x] \times z =$$

$$= 0 \,\&\, y \times x \neq 0 \to x \times z = 0))) \,\&\, h \times x = x \,\&\, x \neq 0.$$

It is clear that if $x = N_k r_{\alpha_k}$, where N_k is a complex number, $N_k \neq 0$, $1 \leq k \leq n$, then $\mathfrak{A}_1(x, c, h)$ is true.

Now suppose that $\mathfrak{A}_1(x, c, h)$ is true. Then there exist complex numbers $N_1, \dots, N_n$ such that $x = N_1 r_{\alpha_1} + \dots + N_n r_{\alpha_n}$. Suppose that among the numbers $N_1, \dots, N_n$ there are two, for example N_{m_1} and N_{m_2}, which are nonzero. Let $s_1 \in [S]$, $(s_1, \alpha_{m_1}) = -i$, and $(s_1, \alpha_j) = 0$ for $j \neq m_1$. Then $s_1 \times c = 0$, $s_1 \times x = N_{m_1} r_{\alpha_{m_1}}$. Hence $s_1 \times x \neq 0$. Now let $(s_2, \alpha_{m_2}) = -i$ and $(s_2, \alpha_j) = 0$ for $j \neq m_2$. Then $s_2 \times x \neq 0$, but $s_2 \times [s_1 \times x] = 0$. The contradiction obtained shows that $x = N r_{\alpha_m}$, where N is a complex number, $N \neq 0$, $1 \leq m \leq n$.

Now consider the formula

$$\mathfrak{B}(z_1, z_2, x_1, x_2) \underset{df}{\Longleftrightarrow} \exists uvw\, ([x_1 \times u] \times w = w \,\&\, [x_2 \times v] \times w =$$

$$= w \,\&\, w \neq 0 \,\&\, [z_1 \times u] \times w = [z_2 \times v] \times w).$$

Let $z_1 = M_1 x_1,\ z_2 = M_2 x_2$, and let $\mathfrak{B}(z_1, z_2, x_1, x_2)$ be true. Then $[z_1 \times u] \times w = M_1 w,\ [z_2 \times v] \times w = M_2 w$, whence $M_1 = M_2$.

Let $x_1 = N_1 r_{\alpha_j},\ x_2 = N_2 r_{\alpha_k},\ N_1 \neq 0,\ N_2 \neq 0,\ z_1 = M x_1,\ z_2 = M x_2$; N_1, N_2, M are complex numbers, $1 \leq j, k \leq n$. We shall show that $\mathfrak{B}(z_1, z_2, x_1, x_2)$ is true. Let $\beta \in \Sigma,\ (\beta, \alpha_j) \neq 0,\ (\beta, \alpha_k) \neq 0$. Put

$$u = -\frac{1}{N_1 \cdot (\beta, \alpha_j)} r_{-\alpha_j}, \qquad v = -\frac{1}{N_2 \cdot (\beta, \alpha_k)} r_{-\alpha_k}, \qquad w = r_\beta.$$

Then $[x_1 \times u] \times w = w,\ [x_2 \times v] \times w = w,\ w \neq 0,\ [z_1 \times u] \times w = [z_2 \times v] \times w$. Thus the formula $\mathfrak{B}(z_1, z_2, x_1, x_2)$ is true for $x_1 = N_1 r_{\alpha_j},\ x_2 = N_2 r_{\alpha_k},\ z_1 = M_1 x_1,\ z_2 = M_2 x_2$, where N_1, N_2, M_1, M_2 are complex numbers, $N_1 \neq 0,\ N_2 \neq 0,\ 1 \leq j, k \leq n$, if and only if $M_1 = M_2$.

Finally, consider an arbitrary partition of the set $\{1, \dots, n\}$ into k parts $A_1, \dots, A_k$. Choose $h_l \in [S]$ so that $(h_l, \alpha_j) = N_l$ if $j \in A_l$, where $N_{l'} \neq N_l$ if $l' \neq l$.

Consider the formula $\mathfrak{B}(h_1 \times x_1, h_1 \times x_2, x_1, x_2)$. It is clear that this formula is true if and only if $x_1 = N_1 r_{\alpha_j},\ x_2 = N_2 r_{\alpha_k}$, where $N_1 \neq 0,\ N_2 \neq 0$, N_1, N_2 are complex numbers, $1 \leq j, k \leq n$, when j and k belong to the same part.

4. Theorem. *Whatever the class M of simple finite-dimensional complex Lie algebras containing Lie algebras of arbitrarily large finite dimensions, there does not exist a recursive set of formulas Φ satisfying the condition*

$$U(L) \subseteq \Phi \subseteq B(M).$$

Proof. One must follow the scheme described in detail in (^{3}) (see also (^{4},^{5})).

Let L_1 be the set of formulas of the first-order predicate calculus without equality, of signature $\langle E_1^{(2)}, E_2^{(2)} \rangle$; $U(L_1)$ the set of identically true closed formulas from L_1; $B(E^2)$ the set of closed formulas from L_1 true on every finite model in which E_1 and E_2 are equivalence relations. From (^{6}) it follows that there does not exist a recursive set Φ satisfying the condition

$$U(L_1) \subseteq \Phi \subseteq B(E^2).$$

To each formula $\mathfrak{C} \in L_1$ we assign a formula $\varphi\mathfrak{C}$. Namely, let

$$\varphi(E_1(x, y)) = \mathfrak{B}(h_1 \times x, h_1 \times y, x, y),$$

$$\varphi(E_2(x, y)) = \mathfrak{B}(h_2 \times x, h_2 \times y, x, y),$$

$$\varphi(\mathfrak{C}_1 \mid \mathfrak{C}_2) = (\varphi\mathfrak{C}_1) \mid (\varphi\mathfrak{C}_2), \qquad \varphi(\forall x\, \mathfrak{C}(x)) = \forall x\, (\mathfrak{A}_1(x, c, h) \to \varphi(\mathfrak{C}(x))),$$

$$\varphi(\exists x\, \mathfrak{C}(x)) = \exists x\, (\mathfrak{C}_1(x, c, h)\ \&\ \varphi(\mathfrak{C}(x))).$$

Let

$$\psi(\mathfrak{C}) \overset{df}{\Longleftrightarrow} \forall c h_1 h_2 h\,(\exists x\, \mathfrak{A}_1(x, c, h) \to \varphi(\mathfrak{C})).$$

It is verified that if $\mathfrak{C} \in TU(L_1)$, then $\psi(\mathfrak{C}) \in TU(L)$, and that if $\psi(\mathfrak{C}) \in B(M)$, then $\mathfrak{C} \in B(E^2)$.

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Note: Figure translations are in progress. See original paper for figures.

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