ON THE METHOD OF AVERAGING

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Abstract

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ON THE METHOD OF AVERAGING

IN SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS

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The method of averaging for systems of differential equations was developed in the works of N. N. Bogolyubov ([1,2]), Yu. A. Mitropolsky ([2,3]), V. M. Volosov ([4]), and other authors.

The present paper is devoted to extending the method of averaging to systems of integro-differential equations of the form

$$\frac{dx}{dt} = \varepsilon X(t,x) + \varepsilon \int_0^t Z(t, x(s), s)\, ds, \tag{1}$$

where $\varepsilon > 0$ is a small parameter; $x = \{x_1, \dots, x_n\}$ is an n-dimensional vector; $X(t,x)$, $Z(t,x,s)$ are real vector-functions, defined and continuous for all t and s from $[0, +\infty)$ and for all $x \in E_n$ (E_n is n-dimensional Euclidean space).

1. Suppose that the limits exist

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x)\, dt = X_0(x), \qquad \lim_{s\to\infty} \frac{1}{s} \int_0^s Z(t,x,s)\, ds = Z_0(t,x).$$

Along with equation (1), consider the averaged equation

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) + \varepsilon \int_0^t Z_0(t, \xi(s))\, ds. \tag{2}$$

Concerning the closeness of the solutions of equations (1) and (2), the following can be proved.

Theorem. *Suppose that the functions $X(t,x)$ and $Z(t,x,s)$ satisfy the following conditions:*

1) For some domain $D \subset E_n$ one can indicate constants M, N, λ, and μ such that, for all $t \geq 0$, $s \geq 0$, and for any points x, x', x'' from D, the inequalities

$$|X(t,x)| \leq M, \qquad |X(t,x') - X(t,x'')| \leq \lambda |x' - x''|,$$

$$|Z(t,x,s)| \leq N, \qquad |Z(t,x',s) - Z(t,x'',s)| \leq \mu |x' - x''|$$

hold.

2) Uniformly with respect to $x \in D$ and t, the limits exist

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x)\,dx = X_0(x), \qquad \lim_{s\to\infty} \frac{1}{s} \int_0^s Z(t,x,s)\,ds = Z_0(t,x).$$

Then, for arbitrarily small positive η, ρ, and for arbitrarily large $L > 0$, one can assign such a positive ε_0 that, if $\xi = \xi(t)$ is a solution of equation (2), defined on the interval $0 \leq t < \infty$ and lying

lying in the domain D together with its ρ-neighborhood, then for $0 \leq \varepsilon < \varepsilon_0$ on the interval $0 < t < Le^{-k}$, $0 < k < 1/2$, the inequality

$$|x(t) - \xi(t)| < \eta, \tag{3}$$

holds, where $x(t)$ is the solution of equation (1) coinciding with $\xi(t)$ at $t = 0$.

Proof. Following the method of N. N. Bogolyubov ([1]), introduce the function

$$u(t,x) = \int_D \Delta_a(x - x') \left\{ \int_0^t [X(\tau,x') - X_0(x')]\,d\tau + \right.$$

$$\left. + \int_0^t \left[\int_0^\tau (Z(\tau,x',s) - Z_0(\tau,x'))\,ds \right] d\tau \right\} dx',$$

where

$$\Delta_a(x) = \begin{cases} A_a \left(1 - \dfrac{|x|^2}{a^2}\right)^2, & |x| \leq a, \\ 0, & |x| > a, \end{cases} \qquad \int_{E_n} \Delta_a(x) = 1.$$

Using the conditions of the theorem, it is not difficult to establish the following inequalities:

$$|u(t,x)| \leq tf(t) + t^2\psi(t), \qquad \left| \frac{\partial u(t,x)}{\partial x} \right| \leq I_a[tf(t) + t^2\psi(t)],$$

$$\left| \frac{\partial u(t,x)}{\partial t} - X(t,x) + X_0(x) - \int_0^t [Z(t,x,s) - Z_0(t,x)]\,ds \right| \leq 2a(\lambda + \mu t),$$

$$I_a = \int_{E_n} \left| \frac{\partial \Delta_a}{\partial x} \right| dx,$$

where $f(t)$ and $\psi(t)$ are monotonically decreasing functions tending to zero as $t \to \infty$.

Let us now form the expression

$$R = \frac{d\bar{x}}{dt} - \varepsilon X(t, \bar{x}) - \varepsilon \int_0^t Z(t, \bar{x}, s)\, ds$$

and compute it for $\bar{x} = \xi(t) + \varepsilon u(t, \xi(t))$, where $\xi(t)$ is the solution of equation (2), defined on the interval $0 \le t < +\infty$ and lying in D with its ρ-neighborhood. Carrying out the corresponding calculations, we find

$$|R(t)| \le 2a\varepsilon(\lambda + \mu t) + 4N\varepsilon t + I_a \varepsilon^2 [t f(t) + t^2 \psi(t)](M + Nt) + \\ + \lambda \varepsilon^2 [t f(t) + t^2 \psi(t)] + \mu \varepsilon^2 [t^2 \alpha(t) + t^3 \beta(t)]. \quad (4)$$

Here $\alpha(t) \to 0,\ \beta(t) \to 0$ as $t \to \infty$.

Let $x(t)$ be the solution of equation (1) coinciding with $\xi(t)$ at $t = 0$. Then on the interval $0 < t < t^*$, $t^* < Le^{-k}$, on which $x(t) \in D$, we have

$$\left| \frac{d(\bar{x} - x)}{dt} \right| \le \lambda \varepsilon |\bar{x} - x| + \mu \varepsilon \int_0^t |\bar{x}(s) - x(s)|\, ds + |R(t)|.$$

Consequently,

$$|\bar{x} - x| \le \left| \frac{1}{k_2 - k_1} \int_0^t |R(\tau)| \left[k_2 e^{k_2 (t-\tau)} - k_1 e^{k_1 (t-\tau)} \right] d\tau \right|,$$

$$k_2 = \frac{\lambda\varepsilon}{2} + \sqrt{\mu\varepsilon + \left(\frac{\lambda\varepsilon}{2}\right)^2}, \qquad k_1 = \frac{\lambda\varepsilon}{2} - \sqrt{\mu\varepsilon + \left(\frac{\lambda\varepsilon}{2}\right)^2}. \quad (5)$$

Now, using the properties of the functions α, β, f, ψ and disposing of the parameters a and ε_0, on the basis of inequalities (4) and (5), we find that on the interval $0 < t < t^*$, $0 \leqslant \varepsilon < \varepsilon_0$, inequality (3) will hold. It is easy to show that $t^* = L\varepsilon^{-k}$. The theorem is proved.

2. If the function $Z(t, x, s)$ is averaged not with respect to s, but with respect to t, i.e., if one assumes the existence of the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T Z(t, x, s)\, dt = Z_{01}(x, s),$$

then we obtain the averaged integro-differential equation of the form

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) + \varepsilon \int_0^t Z_{01}(\xi(s), s)\, ds,$$

which reduces to the system of differential equations

$$\frac{d\xi}{dt} = \eta, \qquad \frac{d\eta}{dt} = \varepsilon \frac{\partial X_0(\xi)}{\partial \xi}\eta + \varepsilon Z_{01}(\xi, t).$$

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CITED LITERATURE

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[2] N. N. Bogolyubov, Yu. A. Mitropolsky, *Asymptotic Methods in the Theory of Nonlinear Oscillations*, 1963.
[3] Yu. A. Mitropolsky, *Problems of the Asymptotic Theory of Nonstationary Oscillations*, 1964.
[4] V. M. Volosov, UMN, **17**, 6, 3 (1962).

Note: Figure translations are in progress. See original paper for figures.