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Abstract

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ON THE SMOOTHNESS OF SOLUTIONS OF DEGENERATE ELLIPTIC AND PARABOLIC EQUATIONS

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Boundary-value problems for a general degenerate elliptic equation of the second order were first considered in the works of G. Fichera (^{1,2}). In (^{3}), for such equations, the existence and uniqueness of a generalized solution of the boundary-value problem posed by G. Fichera were proved. Below we shall establish, by simple means, a number of theorems on the smoothness of the generalized solution of the boundary-value problem constructed in (^{3}); and we shall also prove the existence of a smooth solution of the Cauchy problem and of the first boundary-value problem for an arbitrary degenerate parabolic equation of the second order with smooth coefficients. Differential properties of solutions of degenerate elliptic and parabolic equations of the second order were studied by methods of probability theory in ($^{4-7}$). Some classes of such equations were studied in (^{8}).

1. We shall use the notation of (^{3}). Let

$$L(u) \equiv a^{ij}u_{x_ix_j} + b^iu_{x_i} + cu = f, \qquad a^{ij}\xi_i\xi_j \geq 0, \quad c < 0^*. \tag{1}$$

In a domain A with boundary Σ, consider the generalized solution of the boundary-value problem

$$L(u) = f \text{ in } A; \qquad u = g \text{ on } \Sigma_2 + \Sigma_3, \tag{2}$$

constructed in (^{3}). Here, as in (^{3}), Σ_3 is the set of points of Σ where $a^{ij}n_in_j \neq 0$; $n = (n_1, \dots, n_m)$ is the vector of the inner normal to the boundary of the domain; $b \equiv (b^i - a^i_{x_j})n_i$; $\Sigma_1, \Sigma_2, \Sigma_0$ are the sets of points of $\Sigma - \Sigma_3$ where, respectively, $b > 0$, $b < 0$, $b = 0$; g is a bounded measurable function. By E we denote the set of points of $\bar{A}$ where the determinant $|a^{ij}| = 0$. We shall say that a function $\psi(x)$ belongs to the class C_k in A if $\psi(x)$ has in A bounded generalized derivatives up to order k, inclusive.

We shall use the results of (^{3}), where it was shown that the generalized solution of the boundary-value problem (2) can be obtained as the weak limit in $L_2(A)$, as $\varepsilon \to 0$, of solutions u_ε of the Dirichlet problem for the elliptic equation

$$L_\varepsilon(u) \equiv \varepsilon\Delta u + L(u) = f \text{ in } A; \quad u = g \text{ on } \Sigma. \tag{3}$$

We shall assume that $a^{ij}\xi_i\xi_j \geq 0$ in the domain $D \supset \bar{A}$ and $a^{ij} \in C_2$ in D. One may suppose that $D = R_m$.

Lemma 1. *Let $a^{ij}\xi_i\xi_j \geq 0$ for all x from R_m and all ξ, and $a^{ij} \in C_2$ in R_m. Then for any function $v \in C_2$*

$$\left(a^{ij}_{x_\rho} v_{x_i x_j}\right)^2 \leq M a^{ij} v_{x_i x_k} v_{x_j x_k}, \tag{4}$$

where M depends only on the second derivatives of the functions a^{ij}.

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* Here and everywhere below summation over repeated indices from 1 to m is assumed.

Proof. For any nonnegative function $\varphi(x)$, defined for all x and belonging to the class C_2, the inequality

$$\varphi_x^2 \leqslant 2\{\max|\varphi_{xx}|\}\varphi \tag{5}$$

holds.

Indeed, if we suppose that (5) is not satisfied at the point x_0, we arrive at a contradiction with the condition $\varphi(x) \geqslant 0$, representing φ in the form of a segment of the Taylor series at the point $x_1 = x_0 - 2\varphi(x_0)/\varphi'(x_0)$.

Let $\bar{x}$ be an arbitrary point of R_m. Make a change of variables $y_s = a_{is}x_i$, where the a_{is} are constants and $\|a_{is}\|$ is an orthonormal matrix such that the matrix $\|a^{ij}a_{is}a_{jk}\|$ at the point $\bar{x}$ has diagonal form.

At the point $\bar{x}$ we have

$$a^{ij} v_{x_i x_k} v_{x_j x_k} = \tilde{a}^{\mu\mu} v_{y_\mu y_\nu} v_{y_\mu y_\nu}, \qquad \left(a^{ij}_{x_\rho} v_{x_i x_j}\right)^2 = \left(\tilde{a}^{\mu\gamma}_{x_\rho} v_{y_\mu y_\gamma}\right)^2. \tag{6}$$

Applying (5) to the functions $\tilde{a}^{\mu\mu}$ and $\tilde{a}^{\mu\mu} + 2\tilde{a}^{\mu\gamma} + \tilde{a}^{\gamma\gamma}$, we obtain that

$$(\tilde{a}^{\mu\mu}_{x_\rho})^2 \leqslant \widetilde{M}\tilde{a}^{\mu\mu} \quad \text{and} \quad (\tilde{a}^{\mu\gamma}_{x_\rho})^2 \leqslant \widetilde{M}(\tilde{a}^{\mu\mu} + \tilde{a}^{\gamma\gamma}),$$

where $\widetilde{M}$ does not depend on $\bar{x}$ and depends only on the second derivatives of a^{ij}. Using these inequalities and the expressions (6), we obtain (4).

Introduce the notation

$$B_k = c + \tfrac{1}{4}Mm + b^k_{x_k} + \tfrac{1}{2}\sum_{\substack{i=1\\ i\neq k}}^{m} |b^i_{x_k}| + \tfrac{1}{2}\sum_{\substack{i=1\\ i\neq k}}^{m} |b^k_{x_i}|, \qquad k = 1, \dots, m, \tag{7}$$

where M is the constant in inequality (4), depending on the second derivatives of a^{ij}.

Lemma 2. Let $\Omega \subset A$; let σ be the boundary of the domain Ω; and let $b^i, c, f \in C_1$ in Ω. Let E^δ be the δ-neighborhood of the set E, and suppose that for some $\delta > 0$ the inequalities

$$\sup_{E^\delta \cap \overline{\Omega}} B_k < 0, \qquad k = 1, \dots, m \tag{8}$$

are satisfied.

Then, for the solutions u_ε of problem (3), the estimate

$$\max_{\Omega} |\operatorname{grad} u_\varepsilon|^2 \leqslant M_1 + \max_{\sigma} |\operatorname{grad} u_\varepsilon|^2, \tag{9}$$

holds, where M_1 does not depend on ε.

Proof. Since $c < 0$, the u_ε are uniformly bounded with respect to ε by virtue of the maximum principle. In the domain $Q = \{E^\delta \cap \Omega\}$ the conditions $\sup B_k < 0$, $k = 1, \dots, m$, are fulfilled. In $\Omega - Q$, equation (3) is uniformly elliptic for $\varepsilon \geqslant 0$, and therefore an estimate of the form (9) in $\Omega - Q$ follows from the known estimates of S. N. Bernstein (^{9}) (see (8), pp. 445-448). In view of this, for the proof of Lemma 2 it is sufficient to obtain the estimate (9) for such domains Ω for which $\sup B_k < 0$ in Ω, $k = 1, \dots, m$.

Differentiating equation (3) with respect to x_k, multiplying it by u^k_x, and summing over k from 1 to m, we obtain an equation for $p = |\operatorname{grad} u_\varepsilon|^2$:

$$\tfrac{1}{2}\varepsilon\Delta p - \varepsilon u_{x_{kx}i}u_{x_{kx}i} + \tfrac{1}{2}a^{ij}p_{x_ix_j} + [-a^{ij}u_{x_{kx}i}u_{x_{kx}j} + a^{ij}_{x_k}u_{x_ix_j}u_{x_k}] +$$

$$+\tfrac{1}{2}b^i p_{x_i} + b^i_{x_k}u_{x_i}u_{x_k} + cp + c_{x_k}uu_{x_k} = f_{x_k}u_{x_k}.$$

Using Lemma 1 to estimate the expression in brackets, and taking into account condition (8), we obtain that the maximum of p, if it is attained inside Ω, is bounded uniformly with respect to ε, and therefore (9) is valid.

Lemma 3. Let σ_1 be a part of the boundary Σ of the domain A; let $g = 0$ in a neighborhood of σ_1; all points of σ_1 are interior for $\Sigma_3 + \Sigma_2$; and let $\sigma_1 \subset A^{(2)}$ (see (^{10}), p. 10). Then, for the solutions u_ε of problem (3), $|\operatorname{grad} u_\varepsilon|$ on σ_1 is bounded uniformly with respect to ε.

The proof of boundedness of the first derivatives of u_ε on σ_1 can

can be carried out by the method that was used to estimate the derivatives of the solution on the boundary in paper (11) (see also (8), p. 449).

Further, we shall assume that the conditions for existence and uniqueness of the generalized solution of problem (2) are fulfilled, $a^{ij} \subset C^{(2+\alpha)}$, $b^i \subset C^{(1+\alpha)}$, $c \subset C^{(\alpha)}$, $0 < \alpha < 1$, $A \subset A^{(2)}$ (see (3)).

Theorem 1. Let $\overline{\Omega} \subset A$, $E \cap \sigma = 0$. Let $f \subset C_1$ in Ω, and suppose that for all k $(k = 1, \dots, m)$ on $E \cap \Omega$ the inequalities $B_k < 0$ are fulfilled. Then the generalized solution u of problem (2) belongs to the class C_1 in Ω.

The proof of Theorem 1 follows from estimate (9) and Schauder estimates (12) (see also (10)) or Bernstein estimates (9) for u_ε in a neighborhood of σ, where equations (3) are uniformly elliptic for $\varepsilon > 0$.

Theorem 2. Let the domain A be such that $\overline{\Sigma_3 + \Sigma_2} \cap \Sigma_1 + \Sigma_0 = 0$, let $f \subset C_1$ in A, $g = 0$. On the set $E \cap \overline{A}$, for all $k = 1, \dots, m$, suppose the inequalities $B_k < 0$ are fulfilled. Then the generalized solution u of problem (2) belongs to the class C_1 in A.

For the proof of Theorem 2, note that the solution u of problem (2) in the domain A can be obtained as the weak limit in $\mathcal{L}_2(A)$, as $\varepsilon \to 0$, of the solutions u_ε of the Dirichlet problem in the extended domain A_1, which contains $A + \Sigma_0 + \Sigma_1$, and whose boundary S_1 contains $\Sigma_3 + \Sigma_2$, for the equation

$$\varepsilon \Delta u + a \Delta u + L(u) = f, \qquad u = 0 \text{ on } S_1, \tag{10}$$

where a is a smooth function, $a = 0$ in A and $a > 0$ in $A_1 - \overline{A}$. The assertion of Theorem 2 follows from Lemmas 2 and 3 as applied to (10).

Theorem 3. Let $\overline{\Omega} \subset A$; let a^{ij}, b^i, c, f belong to C_k in Ω, $\sigma \cap E = 0$. Suppose that for any vector $s^l = (s_1, \dots, s_l)$, where $s_i = 1, \dots, m$ and $1 \le l \le k$, on the set $E \cap \overline{\Omega}$ the inequalities $B_{s^l} < 0$ are fulfilled, where

$$\begin{aligned} B_{s^l} \equiv c + \frac{1}{4} M l^2 m + \sum_{i=1}^{l} b^{s_i}_{x_{s_i}} + \sum_{\substack{i,j=1 \\ i \neq j}}^{l} a^{s_i s_j}_{x_{s_i} x_{s_j}} + \frac{1}{2} \sum_{\substack{\rho=1 \\ i \neq s_\rho}}^{l} \sum_{i=1}^{m} \left| b^i_{x_{s_\rho}} \right| \\ + \frac{1}{2} \sum_{\substack{\rho=1 \\ \rho \neq s_i}}^{m} \sum_{i=1}^{l} \left| b^{s_i}_{x_\rho} \right| + \frac{1}{2} \sum_{\substack{\rho,\mu=1 \\ (i,j) \neq (s_\rho, s_\mu),\ \rho < \mu}}^{l} \sum_{i,j=1}^{m} \left| a^{ij}_{x_{s_\rho} x_{s_\mu}} \right| \\ + \frac{1}{2} \sum_{\substack{\rho,\mu=1 \\ (\rho,\mu) \neq (s_i, s_j),\ i \neq j}}^{m} \sum_{i,j=1}^{l} \left| a^{s_i s_j}_{x_\rho x_\mu} \right| . \end{aligned} \tag{11}$$

Then the generalized solution of problem (2) belongs to the class C_k in Ω.

Assuming that we have estimated the derivatives of u_ε of order $k-1$, we form the equation for the sum of squares of all derivatives of u_ε of order k. Using Lemma 1 for v equal to derivatives of order $(k-1)$ of u_ε, and taking into account the condition $B_{s^k} > 0$, as in the case $k = 1$, we obtain that at the maximum point, if it is attained inside Ω, this sum of squares of derivatives is uniformly bounded with respect to ε. To estimate the derivatives on the boundary σ we use Schauder estimates (^{12}) or Bernstein estimates (^{9}).

Note that the condition $B_k < 0$ and the condition $B_{s^l} < 0$ are always fulfilled if $-c$ is sufficiently large.

Theorem 4. Let A be such that $\Sigma = \Sigma_3 + \Sigma_1 + \Sigma_0$ and $\Sigma_3 \cap \Sigma_1 + \Sigma_0 = 0$. Let $g = 0$; $\Sigma_3 \subset A^{(k+1)}$; $a^{ij}, b^i, c, f \subset C_k$ in A. Suppose that for any vector $s^l = (s_1, \dots, s_l)$, where $s_i = 1, \dots, m$, and for $1 \leq l \leq k$, on the set $E \cap \overline{A}$ the inequalities $B_{s^l} < 0$ are fulfilled, where B_{s^l} is the function (11). Then the generalized solution u of problem (2) belongs to the class C_k in A.

Theorem 4 is proved like Theorem 3. The difference is that, for estimating the derivatives of the solutions u_ε of equation (10) on the boundary S_1, one uses a method analogous to that applied in (^{11}). A detailed proof of these estimates of higher derivatives on the boundary is given in paper (^{8}), pp. 458–460.

It is easy to construct examples showing that conditions of the form $B_s > 0$ or $B_{s_l} < 0$, $1 \leq l \leq k$, in Theorems 1-4 are essential.

2. Consider the Cauchy problem in the domain $G\{0 < t < T,\ x \subset R_m\}$ for the equation

$$L(u) - u_t \equiv a^{ij}(t,x)u_{x_i x_j} + b^i(t,x)u_{x_i} + c(t,x)u - u_t = f(t,x). \qquad (12)$$

with the condition

$$u(0,x) = u_0(x), \qquad (13)$$

where $a^{ij}\xi_i\xi_j \geq 0$ in G; R_m is the space of $(x_1, \dots, x_m)$. One can construct a generalized solution of the Cauchy problem (12), (13) and prove its uniqueness analogously to the way this was done for the boundary-value problem in [3]. This solution may be obtained as the limit, as $\varepsilon \to 0$, of solutions of the Cauchy problem in G for the equation

$$\varepsilon \Delta u + L(u) - u_t = f_\varepsilon, \qquad \varepsilon > 0, \qquad (14)$$

with the condition $u(0,x) = u_0^\varepsilon(x)$, where $u_0^\varepsilon(x) \to u_0(x)$ and $f_\varepsilon \to f$ in the corresponding norm as $\varepsilon \to 0$.

We shall say that $\psi \subset C_k$ in G if $\psi(t,x)$ has bounded generalized derivatives with respect to x up to order k inclusive in G; $\Phi \subset \overline{W}_2^k(G)$, if all derivatives of Φ with respect to x up to order k inclusive belong to $L_2(G)$.

Theorem 5. Let a^{ij}, b^i, $c \subset C_k$ in G; $a^{ij} \subset C_2$, $f \subset \overline{W}_2^k(G)$; $u_0(x) \subset W_2^k(R_m)$. Then the solution $u(t,x)$ of the Cauchy problem (12), (13) belongs to $\overline{W}_2^k(G)$.

The proof of this theorem is based on Lemma 1 and on the assumption that the coefficient c in (14) may be considered negative and sufficiently large in absolute value.

Theorem 6. Let $u_0(x) \subset C_k$ in R_m; a^{ij}, b^i, c, $f \subset C_k$ in G ($a^{ij} \subset C_2$ in G, if $k = 1$). Then the solution of the Cauchy problem (12), (13) belongs to the class C_k in G.

This theorem is proved in exactly the same way as Lemma 2, taking into account that $-c$ can be made arbitrarily large after the transformation $u = ve^{\alpha t}$.

Analogous smoothness theorems can be proved for the generalized solution of the first boundary-value problem for equation (12), and also for the boundary-value problem for equation (1) in a domain with piecewise-smooth boundary.

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Note: Figure translations are in progress. See original paper for figures.

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