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Abstract

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MATHEMATICS

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OPTIMAL FORMULAS OF MECHANICAL CUBATURES WITH NODES AT POINTS OF REGULAR LATTICES

In previous notes (¹⁻⁴) we investigated the norm of the error functional in $L_2^{(m)*}$ for cubature formulas for finite functions with constant coefficients and nodes situated in a regular lattice, and also for formulas with a regular boundary layer in domains with a sufficiently smooth boundary. In the present note we establish that the optimal coefficients of cubature formulas with a regular lattice of nodes have the same principal term in the norm of the error functional as do formulas with a regular boundary layer.

By Babushka's theorem (⁶), the optimal coefficients of cubature formulas with error functional

$$l(x) = \mathcal{E}_\Omega(x) - \sum_{\beta} C_{\beta} \delta(x - hH\beta)$$

are characterized by the fact that the solutions from $L_2^{(m)}$ of the equation $\Delta^m u_0 = (-1)^m l_0(x)$ take at the points $hH\gamma$ values coinciding with the values of a certain polynomial of degree $m - 1$. In other words,

$$u_0(hH\beta) = (l_0(x) * G(x))|_{x=hH\beta} = P(hH\beta).$$

Let us also consider a cubature formula with the same nodes and with a regular boundary layer of order $2m + 2$

$$l_1(x) = \sum_{\beta'} l_{\beta'}^{(1)} \left(\frac{x}{h} - H\beta' \right) = \mathcal{E}_\Omega(x) - \sum_{\beta} C_{\beta} \delta(x - hH\beta).$$

The square of the norm of the error of the cubature formula is a second-degree polynomial in the coefficients C_{β}

$$\begin{aligned} \|l(x)\|_{L_2^{(m)*}}^2 &= (l(x) * G(x) * l(-x))|_{x=0} = \Psi(C) = \\ &= (\mathcal{E}_\Omega(x) * G(x) * \mathcal{E}_\Omega(-x))|_{x=0} - 2 \sum_{\beta} C_{\beta} (\mathcal{E}_\Omega(x) * G(x - hH\beta))|_{x=0} + \\ &\quad + \sum_{\beta} \sum_{\beta'} G(hH(\beta - \beta')) C_{\beta} C_{\beta'}. \end{aligned}$$

The difference $\Psi(C) - \Psi(C^{(0)})$ is, consequently, in its turn a second-degree polynomial in the differences $C_{\beta} - C_{\beta}^{(0)}$. It can be proved that this difference is expressed simply by the quadratic form

$$\Psi(C) - \Psi(C^{(0)}) = \sum_{\beta} \sum_{\beta'} G(hH(\beta - \beta')) (C_{\beta} - C_{\beta}^{(0)}) (C_{\beta'} - C_{\beta'}^{(0)}).$$

Denoting

$$\rho[\beta] = 0, \quad hH\beta \notin \Omega; \quad \rho[\beta] = C_{\beta} - C_{\beta}^{(0)}, \quad hH\beta \in \Omega, \quad (1)$$

we obtain the form already studied by us earlier in note ⁽⁵⁾:

$$\Psi(C) - \Psi(C^{(0)}) = \Delta_{hH}(U_{hH}[\beta], U_{hH}[\beta]),$$

where

$$U_{hH}[\beta] = \rho[\beta] * G_{hH}[\beta]$$

and associated with the difference generalization of the polyharmonic potential.

Theorem 1. The values of the difference potential $\rho[\beta] * G_{hH}[\beta] = U_{hH}[\beta]$ at interior points of the domain Ω differ by a polynomial of degree $m - 1$ from the values $u(hH\beta)$, where

$$u(x) = l(x) * G(x). \quad (2)$$

Proof. For the difference $u(x) - u_0(x)$ we have $u(x) - u_0(x) = (l(x) - l_0(x)) * G(x) = \sum (C_{\beta} - C_{\beta}^{(0)}) \delta(x - hH\beta) * G(x)$, and, hence, $u(hH\beta) - u_0(hH\beta) = \rho[\beta] * G_{hH}[\beta]$, whence from (1) it follows that $U_{hH}[\beta] - u(hH\beta) = -u_0(hH\beta) = -P(hH\beta)$ for $hH\beta \in \Omega$. Theorem 1 is proved.

Theorem 2. The potential $U_{hH}[\beta]$ gives the absolute minimum of the functional $\Delta_{hH}(\varphi[\beta], \varphi[\beta])$ among all functions $\varphi[\beta]$ belonging to $l_2^{(m)}$ and coinciding at the points $hH\beta \in \Omega$ with the values $U_{hH}[\beta]$,

$$\Delta_{hH}(U_{hH}[\beta], U_{hH}[\beta]) \leq \Delta_{hH}(\varphi[\beta], \varphi[\beta]),$$

if $\varphi[\beta] = U_{hH}[\beta]$ for $hH\beta \in \Omega$.

The proof is based on several lemmas. First it is necessary to extend the scalar products Δ_{hH} to a somewhat broader space, and then to use Green's formula for its transformation. Let $M_p^{(l)}(N)$ be the space of functions $\varphi(x)$ defined in E_n with norm determined by the equality:

$$\|\varphi\|_{M_p^{(l)}(N)} = \left\{ \int_{-\infty}^{+\infty} \left[\max_{|y_j - z_j| < N} \sum_{|\alpha|=m} (D^\alpha \varphi(y))^2 \right]^{p/2} dz \right\}^{1/p}.$$

Lemma 1. The spaces $M_p^{(l)}(N)$ for different values are equivalent to one another.

The proof is based on two inequalities:

$$\|\varphi\|_{M_p^{(l)}(N_1)}^p \geq \|\varphi\|_{M_p^{(l)}(N_2)}^p \quad \text{for } N_1 \geq N_2; \quad (3)$$

$$\|\varphi\|_{M_p^{(l)}((2k+1)N)}^p \leq (2k+1)^n \|\varphi\|_{M_p^{(l)}(N)}^p, \quad (4)$$

which are established elementarily. From (3) and (4) the lemma follows almost immediately.

Lemma 2. Let $\varphi[\beta] \in l$, i.e., let it have differences of order l , summable with degree p over the whole space. Then there exists a function $\varphi(x) \in M_p^{(l)}(N)$, coinciding with $\varphi[\beta]$ at all points $x = hH\beta$, $\varphi(hH\beta) = \varphi[\beta]$, and such that

$$\|\varphi(x)\|_{M_p^{(l)}(N)} \leq K \|\varphi[\beta]\|_{l_p^{(l)}}.$$

The proof does not differ from the analogous assertion for $L_p^{(l)}$. V. S. Ryabenkii and A. F. Filippov [7] constructed an interpolation operator $\Pi\varphi[\beta]$ which assigns to any function $\varphi[\beta]$ given on the integer lattice β a certain function $\varphi(x)$ possessing the properties: 1) $\varphi(\beta) = \varphi[\beta]$, 2) $|D^\alpha \varphi| \leq K \max_{|hN\beta-x|<Lh} \Delta^\alpha \varphi[\beta]$. The analysis shows that the Ryabenkii-Filippov operator gives the interpolation function needed by us.

Lemma 3. The values of any function $\varphi(x) \in M_p^{(l)}(N)$ at the points $hH\beta$: $\varphi[\beta] = \varphi(hH\beta)$, constitute an element of the space $l_p^{(l)}$.

The proof of Lemma 3 is carried out by means of estimates and is based on the integral representation of differences through derivatives.

Theorem 3. In the space $M_p^{(l)}$, finite functions form a dense set.

The proof of this theorem is rather laborious, but it differs little from the proof of the theorem on the density of finite functions in L (see ^(8,9)), and therefore we shall not reproduce the corresponding line of reasoning here.

Theorem 4. In the space $l_p^{(l)}$, the finite functions form a dense set.

The proof of Theorem 4 follows from Theorem 3, and also from the equivalence of $l_p^{(l)}$ and $M_p^{(l)}$, when a correspondence is established between them by the methods indicated in Lemmas 2 and 3.

Lemma 4 (Green's formula). The formula

$$\Delta_{hH}(u[\beta], v[\beta]) = (u[\beta], L_{hH}[\beta] * v[\beta]), \quad (5)$$

which serves as the definition of the scalar product Δ_{hH} , in the case where both functions $u[\beta]$, $v[\beta]$ decrease exponentially at infinity, remains valid if one of the functions $u[\beta]$ or $v[\beta]$ is an arbitrary element of $l_2^{(m)}$, and the other is finite.

The proof of this lemma follows from Theorem 4 and the equivalence of the scalar product $\Delta_{hH}(\varphi, \psi)$ to the square of the norm of the function $u[\beta]$ in $l_2^{(m)}$ (see (5)).

Lemma 5. If the density $\rho[\beta]$ is orthogonal to all polynomials of degree $m - 1$, $(\rho[\beta], (hH\beta)^\alpha) = 0$, $|\alpha| \leq m - 1$, then the function $U_{hH}[\beta] = \rho[\beta] * G_{hH}[\beta]$ is an element of $l_2^{(m)}$.

The proof is carried out in the same way as the corresponding proof for the convolution $l(x) * G(x)$, given in ^(1,10).

We now indicate the method of proving Theorem 2. Let $u[\beta] = 0$ for $hH\beta \in \Omega$ and $L_{hH} * v[\beta] = 0$ for $hH\beta \notin \Omega$, with $u \in l_2^{(m)}$ and $v \in l_2^{(m)}$. Then $\Delta_{hH}(u[\beta], v[\beta]) = 0$, as follows easily from (5).

Put $\varphi[\beta] - U[\beta] = u[\beta]$. Formula (5) is valid for the product $(U[\beta], u[\beta])$, since $u[\beta] = 0$ for $hH\beta \in \Omega$, by assumption, and $L_{hH}[\beta] * U[\beta] = 0$; $hH\beta \notin \Omega$, because the operator $L_{hH}[\beta]$ is inverse to convolution with $G_{hH}[\beta]$.

Using Lemma 5, we shall have

$$\begin{aligned} \Delta_{hH}(\varphi[\beta], \varphi[\beta]) &= \Delta_{hH}(\varphi[\beta] - U[\beta], \varphi[\beta] - U[\beta]) \\ &\quad + 2\Delta_{hH}(\varphi[\beta] - U[\beta], U[\beta]) + \Delta_{hH}(U[\beta], U[\beta]) = \\ &= \Delta_{hH}(\varphi[\beta] - U[\beta], \varphi[\beta] - U[\beta]) + \Delta_{hH}(U[\beta], U[\beta]), \end{aligned}$$

whence Theorem 5 follows at once.

Theorem 5. *The deviation of the square of the norm of the error functional $l(x)$ from the minimal one satisfies the inequality*

$$\|l(x)\|^2 - \|l_0(x)\|^2 \leq \Delta_{hH}(u(hH\beta), u(hH\beta)), \quad (6)$$

where $u(x)$ is defined by formula (2).

This theorem follows from Theorems 1 and 2.

In order to complete our estimate, it remains to prove the last theorem.

Theorem 6. *The inequality*

$$\Delta_{hH}(u(hH\beta), u(hH\beta)) \leq Kh^{2m+1} \quad (7)$$

is valid.

Before giving the idea of the proof, let us note that from Theorem 6 the result we need will follow. Indeed, for the functional $l(x)$ with a regular boundary layer the formula is valid (see (2))

$$\|l(x)\|_{L_2^{(m)*}}^2 = (h/2\pi)^{2m} \zeta(H^{-1} | 2m) |\Omega| + O(h^{2m+1}). \quad (8)$$

Comparing (8), (7), and (6), we see that for $l_0(x)$ the same estimate is valid:

$$\|l_0(x)\|_{L_2^{(m)*}}^2 = (h/2\pi)^{2m} \zeta(H^{-1} | 2m) |\Omega| + O(h^{2m+1}).$$

Hence

$$\|l_0(x)\|_{L_2^{(m)*}} = (h/2\pi)^m \sqrt{\zeta(H^{-1} | 2m)} \sqrt{|\Omega|} + O(h^{m+1}).$$

The proof of Theorem 6 is based on the representation $u(x) = u_0(x) - w(x)$, where $u_0(x)$ is the periodic solution of the equation $\Delta^m u = (-1)^m (1 - \Phi_0(h^{-1}H^{-1}x))$, given in (2). Here $w(x) = G(x) * l_2(x)$, and $l_2(x)$ is the error functional with a regular boundary layer for the exterior of the domain Ω :

$$l_2(x) = 1 - \Phi_0(h^{-1}H^{-1}x) - l_1(x) = \sum_{\beta \in \Omega} l_{\beta}^{(2)} \left(\frac{x}{h} - H\beta \right),$$

where $(l_\beta^{(2)}(x), x^\alpha) = 0$ for $|\alpha| < 2m + 2$, and all $l_\beta^{(2)}$ for $d(hH\beta, \Omega) > Lh$ are equal to $l_0^{(2)}(x)$.

Let $u_{\beta_1}^{(j)}(x) = G(x) * l_{\beta_1}^{(j)}(x)$, $j = 1, 2$. Computing $\Delta_{hH}(u(hH\beta), u(hH\beta))$, we obtain

$$\begin{aligned} \Delta_{hH}(u(hH\beta), u(hH\beta)) &= \\ &= (u_1(hH\beta), L_{hH}[\beta] * u_0(hH\beta)) - (w(hH\beta), u_1(hH\beta)). \end{aligned} \quad (9)$$

The first term on the right-hand side of (9) is equal to zero. Indeed, $u_0(hH\beta)$, obviously, is constant, and the convolution operator with $L_{hH}[\beta]$ is orthogonal to all polynomials of degree below m . We transform the second term into a sum, similarly to what was done in (4):

$$\Delta_{hH}(w(hH\beta), u_1(hH\beta)) = \sum_{\beta_1, \beta_2} \Delta_{hH}(u_{\beta_1}^{(1)}(hH\beta), u_{\beta_2}^{(2)}(hH\beta)). \quad (10)$$

Expanding further the right-hand side of (10),

$$\Delta_{hH}(u_{\beta_1}^{(1)}(hH\beta), u_{\beta_2}^{(2)}(hH\beta)) = \left(u_{\beta_1}^{(1)}(-hH\beta) * L_{hH}[\beta] * (G(x) * l_{\beta_2}^{(2)}(x)) \Big|_{x=hH\beta} \right),$$

we obtain

$$\begin{aligned} v_{\beta_2}[\beta] &= L_{hH}[\beta] * [(G(x) * l_{\beta_2}^{(2)}(x)) \Big|_{x=hH\beta}] = \\ &= \sum_{\beta'} L_{hH}(\beta - \beta') \int G(hH\beta' - x) l_{\beta_2}^{(2)}\left(\frac{x}{h}\right) dx = \\ &= \int \left[\sum_{\beta'} L_{hH}(\beta - \beta') G(hH\beta' - x) \right] l_{\beta_2}^{(2)}\left(\frac{x}{h}\right) dx. \end{aligned}$$

The function

$$\sum_{\beta'} L_{hH}(\beta - \beta') G(hH\beta' - x) = \tau(x),$$

as was shown in (5), decreases exponentially at infinity. Hence $v_{\beta_2}[\beta] \leq e^{-\eta|\beta|}$. Further, by assumption, $u_{\beta_1}^{(1)}[\beta]$ decreases and $u_{\beta_1}^{(1)}(x) \leq Kh^{2m+2n+2}/(h^2 + x^2)^{(n+1)/2}$. Thus

$$\Delta_{hH}(u_{\beta_1}[\beta], v_{\beta}[\beta]) \leq \frac{Kh^{2m+2n+2}}{(h^2 + x^2)^{(n+1)/2}}.$$

An integral estimate of the sum

$$\sum_{\beta_1, \beta_2} \Delta_{hH}(u_{\beta_1}[\beta], v_{\beta_2}[\beta]),$$

similar to that carried out in (5), gives the proof of Theorem 5.

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Note: Figure translations are in progress. See original paper for figures.

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