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Abstract

Full Text

MATHEMATICS

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ON A DIFFERENCE ANALOGUE OF THE POLYHARMONIC EQUATION

For convenience we shall regard γ as a column vector in the n-dimensional space E_n. Let β be a column vector with integer elements. Generalized functions of the form $\psi(x) = \sum \varphi[\beta]\delta(x-\beta)$ will be called **lattice** functions, and functions $\varphi[\beta]$, defined on integer vectors, **discrete**. We shall denote the set of lattice functions by P, and the set of discrete functions by R. Together with any continuous function $\varphi(x)$ we shall consider the corresponding lattice and discrete functions

$$\begin{aligned}\Omega^{CR}[\beta \mid \varphi] = \varphi(\beta); \qquad \Omega^{CP}(x \mid \varphi) = \sum \varphi(\beta)\delta(x-\beta) = \\ = \varphi(x)\Phi_0(x) = \psi(x),\end{aligned} \tag{1}$$

where $\Phi_0(x) = \sum \delta(x-\beta)$. We shall also write relation (1) in the form

$$\psi(x) = \Omega^{RP}(x \mid \varphi) \quad \text{or} \quad \varphi[\beta] = \Omega^{PR}[\beta \mid \varphi]. \tag{2}$$

Let us introduce also the operations

$$\Omega^{PC}(x \mid \varphi), \qquad \Omega^{RC}(x \mid \varphi), \tag{3}$$

where

$$\Omega^{PC} = (\Omega^{CP})^{-1}, \qquad \Omega^{RC} = (\Omega^{CR})^{-1}. \tag{4}$$

The operations (3) and (4) are not single-valued. For Ω^{PC} one can write the explicit expression $\Omega^{PC}(x \mid \varphi) = \varphi(x) * \Lambda(x)$, where $\Lambda(x)$ is an arbitrary solution of the equation $\Lambda(x) \cdot \Phi_0(x) = \delta(x)$, i.e. a function taking the values $\Lambda(\beta) = 1$, $\beta = 0$; $\Lambda(\beta) = 0$, $\beta \neq 0$. Here, of course, $\Lambda(x)$ must be a multiplier of the function $\varphi(x)$.

We give a table for the spaces to which the products and convolutions of two functions belonging to C or P belong.

$$(\varphi, \psi) = C \in E \tag{5}$$

exists if φ and ψ do not both simultaneously belong to P.

a) $\varphi \cdot \psi = \chi$

$\varphi \backslash \psi$	C	P
C	C	P
P	P	

b) $\varphi * \psi = \chi$

$\varphi \backslash \psi$	C	P
C	C	C
P	C	P

(6)

Operations on elements of R are obvious. (Convolution, of course, is possible not for every pair, but only for sufficiently rapidly decreasing ones.) Operators Ω mapping the spaces C, P, and R preserve the relations (5) and (6a) in all cases, and the validity of (6b) everywhere except in the case $\varphi \in C$, $\psi \in C$.

Let us also consider the space Φ of sufficiently rapidly decreasing functions (for example, finite functions), the space of periodic functions Π

with integer periods, and the space T of functions defined on the torus Ω_0, obtained by identifying all points of E_n whose coordinates differ by integers. Introduce the operators $\Xi^{\Phi\Pi}$, $\Xi^{\Pi\Phi}$, $\Xi^{T\Pi}$, $\Xi^{\Pi T}$, $\Xi^{\Phi T}$, $\Xi^{T\Phi}$. The operators $\Xi^{\Pi T}$ and $\Xi^{T\Pi}$ take a periodic function $\varphi(P)$, defined in E_n, into a function defined on the torus with the same values, and conversely. The operator $\Xi^{\Phi\Pi}$ assigns to $\varphi(P) \in \Phi$ the function

$$\psi(P) = \Xi^{\Phi\Pi}(p|\varphi) = \sum_{\gamma} \varphi(p-\gamma) = \varphi(p) * \Phi_0(p),$$

where the vector γ has integer coordinates. Further,

$$\Xi^{\Phi T} = \Xi^{\Phi\Pi}\Xi^{\Pi T}.$$

The scalar product of functions $\varphi \in \Phi, \Pi$ and $\psi \in \Phi, \Pi$, $(\varphi, \psi) = C \in E_1$, is meaningful if φ and ψ do not both belong to Π.

The tables for the spaces in which the basic binary operators—multiplication, convolution, and scalar product—will lie have the form

a) $\varphi \cdot \psi = \chi$

$\varphi\backslash\psi$	Φ	Π
Φ	Φ	Φ
Π	Φ	Π

b) $\varphi * \psi = \chi$

$\varphi\backslash\psi$	Φ	Π
Φ	Φ	Π
Π	Π	

The product of elements of Π is, of course, possible in the case when φ or ψ are not generalized functions. In this case special qualifications are needed, which we shall not make for brevity. We shall not consider it. The operators Ξ, mapping the spaces Φ, Π, and P, preserve the binary operations and the scalar product $(\varphi, \psi) = \int \varphi\psi \, dp$, except for the case $\varphi \cdot \psi = \chi$ or $(\varphi, \psi) = C$, when $\varphi \in \Phi$ and $\psi \in \Phi$.

The Fourier transforms of functions and generalized functions defined in E_n, for arbitrary elements $f(x) \in L_2^{(m)}$, will be written in the form

$$\tilde{f}(x) = \tilde{f}(p) = \int_{-\infty}^{+\infty} e^{2\pi ipx} f(x)\, dx; \qquad \hat{f}(x) = \hat{f}(p) = \int_{-\infty}^{+\infty} e^{-2\pi ipx} f(x)\, dx. \tag{7}$$

The Fourier transform in the form (7) is unitary. By weak continuity it is, as is well known, extended to the space of generalized functions. The theorems proved in all courses are valid:

Theorem 1. The formulas

$$\hat{\tilde{f}}(x) = \tilde{\hat{f}}(x) = f(x); \qquad \tilde{\tilde{f}}(x) = \hat{\hat{f}}(x) = f(-x).$$

Theorem 2 (Parseval formula)

$$(\tilde{f}(p), \widetilde{\varphi}(p)) = (f(x), \varphi(x)).$$

This theorem makes it possible to construct Fourier transforms for generalized functions.

Theorem 3. There is a duality between the operations of multiplication and convolution:

$$\widetilde{f(x) \cdot \varphi(x)} = \tilde{f}(p) * \widetilde{\varphi}(p); \qquad \widehat{f(x) \cdot \varphi(x)} = \hat{f}(p) * \widehat{\varphi}(p).$$

The Fourier transform of the simplest functions is given by the formulas:

$$\tilde{\delta}(x) = 1; \qquad \tilde{1} = \delta(p); \qquad \widetilde{e^{-\pi x^2}} = e^{-\pi p^2}; \qquad \widetilde{\Phi}_0(x) = \Phi_0(p); \qquad \widetilde{D^\alpha}(x) = (2\pi i p)^\alpha,$$

where $D^\alpha(x)$ is the generalized function whose convolution gives the derivative of order α. The Fourier transform $\Phi_0(x)$ is the so-called Poisson formula (see (1)).

The Fourier transform takes periodic functions into lattice functions and conversely. It extends to discrete functions if one sets $\widetilde{\varphi}[\beta] = \widetilde{\varphi}(p) \equiv \Omega^{PT}(\Omega^{RP}(x/\varphi))$.

At the same time Theorems 1, 2, and 3 remain valid.

Theorem 4. *The Fourier transform maps the spaces C, P, and R one-to-one onto Φ, Π, and T, respectively. The correspondences established by the operators Ω pass into analogous correspondences established by the operators Ξ.*

All the theorems formulated are carried over to lattice functions having δ-type singularities at the lattice points $A\beta$, to which there correspond periodic functions $\varphi(p)$ with periods γA^{-1}. Let us give the corresponding definitions.

Let P_A be the space of lattice functions of the form $\psi(x) = \sum_\beta \varphi[\beta]\delta(x - A\beta)$. The spaces C, P_A, and R are mapped into one another with the aid of the operators $\Omega_A^{CP}, \Omega_A^{PC}, \Omega_A^{PR}, \Omega_A^{RP}, \Omega_A^{CR}, \Omega_A^{RC}$, analogous to the operators Ω considered above. These mappings again preserve the simplest binary operations, as in the tables indicated above. Here it is assumed that $\Omega_A^{CP}(x, \varphi) = \varphi(x)|A|^{-1}\Phi_0(A^{-1}x)$, $\Omega_A^{CR}[\beta|\varphi] = \varphi(A\beta)$. The remaining Ω_A are constructed analogously. All Ω_A are expressed elementarily in terms of the corresponding Ω.

Let us also consider Π_B, the space of periodic functions $\varphi(p)$ with periods γB, where B is a nonsingular matrix, and the corresponding space T_B. $\varphi(p) = \varphi(p + \gamma B)$. Between Φ, Π_B, and T_B act the mapping operators Ξ_B. For $\Xi_B^{\Phi\Pi}$ we shall use the formula $\Xi_B^{\Phi\Pi}(p|\varphi) = |B| \sum_\gamma \varphi(p - \gamma B) = \varphi(p) * \Phi_0(pB^{-1})$. The remaining ones are obvious or are obtained from these. All Ξ_B are expressed elementarily in terms of the corresponding Ξ.

Theorem 5. *The Fourier transform establishes a one-to-one correspondence between C, P_A, R and $\Phi, \Pi_{A^{-1}}, T_{A^{-1}}$. This correspondence takes the local operators Ω_A into the operators $\Xi_{A^{-1}}$, and conversely.*

Theorem 6. *Introduce the scalar product and convolution in T_B by the formulas*

$$\varphi(p) * \psi(p) = |B|^{-1} \int \varphi(p - q)\psi(q)\, dq, \qquad (\varphi(p), \psi(p)) = |B|^{-1} \int \varphi(p)\psi(p)\, dp;$$

under this definition the scalar product is invariant under the Fourier transform, while the product passes into convolution and conversely.

The polyharmonic equation

$$\Delta^m u = f \tag{8}$$

with inverse operator $G(x) * f(x) = u(x)$, where

$$G(x) = \varkappa |x|^{2m-n},\ n \text{ odd}; \qquad G(x) = \varkappa |x|^{2m-n} \ln |x|,\ n \text{ even}, \tag{9}$$

is often studied using the generalized scalar product

$$D(\varphi, \psi) = \int \sum_{|\alpha|=m} D^\alpha \varphi D^\alpha \psi \, dx = (-1)^m \int \varphi \Delta^m \psi \, dx = (-1)^m \int \psi \Delta^m \varphi \, dx.$$

For discrete functions $\varphi[\beta]$ there exist various ways of constructing analogues of Δ^m, G, and $D(\varphi, \psi)$. Often one encounters $\hat{\Delta}^m[\beta]$, where $\hat{\Delta} * \varphi = [\sum_j (\delta[\beta + i_j] + \delta[\beta - i_j] - 2\delta[\beta])] * \varphi[\beta] \equiv \sum_j [\varphi[\beta + i_j] + \varphi[\beta - i_j]] - 2n\varphi[\beta]$. The operator inverse to $\hat{\Delta}^m$ is also a convolution

with some discrete function that asymptotically behaves like $G[\beta]$, while the role of $D(\varphi, \psi)$ is played by the sum

$$\Delta(\varphi, \psi) = \sum_\beta (\hat{\Delta}_\alpha \varphi[\beta], \hat{\Delta}_\alpha \psi[\beta]).$$

We shall give here one more generalization of these notions. Let $G(x)$ be the elementary solution of equation (8), given by (9). Let

$$G_{hH}(x) = \Omega^{CP}_{hH}(x \mid G); \quad G_{hH}[\beta] = \Omega^{CR}_{hH}[\beta \mid G] = G(hH\beta), \tag{10}$$

where $|H| = 1$. We shall call the convolution $G_{hH}[\beta] * \rho[\beta] = U[\beta]$ a **discrete potential**. This is a natural generalization of the convolution $G(x) * \rho(x)$. For the inverse operator $L_{hH}[\beta]$ we obtain

$$G_{hH}[\beta] * L_{hH}[\beta] = \delta[\beta].$$

Passing in equality (10) to lattice functions and using the Fourier transform, we can compute $\widetilde{L}_{hH}(p)$, which turns out to be equal to

$$\widetilde{L}_{hH}(p) = \left[\frac{1}{(2\pi)^{2m}} \sum_\gamma \frac{h^{-n}}{\left\{ \sum_j \left| p_j - (\gamma h^{-1} H^{-1})_j \right|^2 \right\}^m} \right]^{-1}. \tag{11}$$

From formula (11) there follows a series of theorems:

Theorem 7. The convolution operator with $L_{hH}[\beta]$ is orthogonal to all monomials of degree lower than $2m$:

$$L_{hH}[\beta] * \beta^\alpha = 0 \quad \text{for } |\alpha| < 2m.$$

Theorem 8. The discrete function $L_{hH}[\beta]$ is representable in the form

$$L_{hH}[\beta] = h^{2m-n} L_H[\beta],$$

where $L_H[\beta]$ decreases exponentially at infinity:

$$L_H[\beta] \leq e^{-\eta|\beta|}.$$

Theorem 9. On the class of functions $\varphi(x)$ continuously differentiable $2m$ times,

$$L_{hH}(x) * \varphi(x) \xrightarrow[h\to 0]{\text{weakly}} h^n \Delta^m \varphi.$$

Theorem 10. The convolution

$$\tau_{hH} = L_{hH}(x) * G(x) = \tau_H(h^{-1}x)$$

decreases exponentially at infinity:

$$|\tau_{hH}(x)| = |\tau_H(h^{-1}x)| \leq K e^{-\eta h^{-1}|x|}.$$

Theorem 11. For any two real discrete functions $\varphi[\beta]$ and $\psi[\beta]$, exponentially decreasing at infinity, the bilinear form

$$\Delta_{hH}(\varphi,\psi) = (\varphi[\beta], L_{hH}[\beta] * \psi[\beta]) = (\varphi[\beta] * L_{hH}[\beta] * \psi[-\beta])_{\beta=0}$$

is symmetric and nonnegative; moreover, we have

$$\Delta_{hH}(\varphi(hH\beta), \psi(hH\beta)) \xrightarrow[h\to 0]{\text{weakly}} D(\varphi,\psi), \qquad m < \frac{\Delta_{hH}(\varphi,\psi)}{\Delta(\varphi,\psi)} < M,$$

where m, M are two constants independent of φ, ψ.

The proof of Theorems 6–10 follows from the fact that the function $\widetilde{L}_{hH}(p)$ is a function analytic and nonnegative for all real p. It is obviously representable in the form

$$\widetilde{L}_{hH}(p) = h^{n-2m} \widetilde{L}_H(ph).$$

Further, at the origin its product by $\widetilde{G}(p)$ is regular and turns into h^n.

The proof of Theorem 11 follows from the fact that the ratio

$$\widetilde{L}_{hH}(p) \Big/ \left(\sum_j \sin^2 \frac{(phH)_j}{2} \right)^m$$

is enclosed between finite limits, and the magnitude of the forms $\Delta(\varphi,\psi)$ and $\Delta_{hH}(\varphi,\psi)$ is expressed through the Fourier transform by the integrals

$$\Delta_{hH}(\varphi,\psi) = \int \widetilde{L}_{hH}(p)\, \varphi(p)\, \overline{\psi}(p)\, dp,$$

$$\Delta(\varphi, \psi) = \int \left(\sum_j \sin^2 \frac{(phH)_j}{2} \right)^m \varphi(p)\, \overline{\psi}(p)\, dp.$$

The finiteness of the ratio of the integrand functions entails the finiteness of the ratio of the integrals.

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Note: Figure translations are in progress. See original paper for figures.