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Abstract

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MATHEMATICS

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AN ELEMENTARY METHOD OF SEPARATING VARIABLES IN EQUATIONS WITH MANY VARIABLES

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In the present note, by separation of variables in the equation

$$F(u_1, \ldots, u_n) = 0, \qquad n \geqslant 4, \tag{1}$$

we mean the representation of this equation in the equivalent form

$$F_1 = F_2, \tag{2}$$

where $F_1 = F_1(u_\alpha, \ldots, u_\delta)$, $F_2 = F_2(u_\varepsilon, \ldots, u_\lambda)$, and $\alpha, \ldots, \delta, \varepsilon, \ldots, \lambda$ is some permutation of the indices $1, \ldots, n$, or, in other words, its reduction to the system of equations

$$F_1 - s = 0,$$

$$F_2 - s = 0, \tag{3}$$

where s is a new auxiliary variable.

Separating variables in the equations of system (3), and then in the equations obtained as a result of this, and proceeding further in the same way, in a number of cases equation (1) can thus be reduced to a system of equations with three variables each.

Conditions for separating variables in equations with four variables were found by E. Goursat ([1],[2]); methods for separating variables in equations with many variables were proposed by O. V. Ermolova ([3]), Ya. Voitovich ([4]), Yu. I. Bogolyubov ([5]), and others. These methods ultimately reduce to integrating differential equations arising from comparing equations (1) and (2), and their realization requires expositions of considerable length.

Separation of variables in equations with many variables has practical significance. The representation of equation (1) by composite nomograms of various kinds, the representation of the dependence determined by this equation by a system of tables with two inputs each ([6]), and also the solution of certain other problems are possible only in those cases when the corresponding separation of variables in the given equation is known. It is therefore quite natural to seek a sufficiently simple method of separating variables. The method presented in the present note arose precisely along these lines.

The representation (2) for equation (1) can be found in the following way. It is assumed that equation (1), in the parallelepiped

$$\bar{u}_i \leqslant u_i \leqslant \bar{\bar{u}}_i, \qquad i = 1, \dots, n,$$

defines functions

$$u_i = \theta_i(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n) \tag{4}$$

and admits a representation (2) in which $F_1(u_\alpha, u_\beta, \dots, u_\delta)$ is a strictly monotone function of one of its arguments when the others are constant, for example, of the argument u_α when

$$u_\beta = u_{\beta_0}, \dots, u_\delta = u_{\delta_0}, \qquad \bar{u}_j \leqslant u_{j_0} \leqslant \bar{\bar{u}}_j, \qquad j = \beta, \dots, \delta. \tag{5}$$

Under these assumptions, for the function $F_1(u_\alpha, u_{\beta_0}, \dots, u_{\delta_0})$ there exists an inverse

$$\widetilde{F}_1 = F_1(u, u_{\beta_0}, \dots, u_{\delta_0}), \tag{6}$$

i.e.

$$\widetilde{F}_1(F_1(u_\alpha, u_{\beta_0}, \dots, u_{\delta_0}), u_{\beta_0}, \dots, u_{\delta_0}) \equiv u_\alpha.$$

If the function (6) is taken of both sides of equation (2), one obtains the equation

$$\widetilde{F}_1(F_1(u_\alpha, \dots, u_\delta), u_{\beta_0}, \dots, u_{\delta_0}) = \widetilde{F}_1(F_2(u_\varepsilon, \dots, u_\lambda), u_{\beta_0}, \dots, u_{\delta_0}), \tag{7}$$

which, like (2), is an equation equivalent to (1), but possessing the property that, for the values of the variables (5), its left-hand side becomes identically equal to u_α, i.e. equation (7), for the values of the variables (5), takes the form

$$u_\alpha = \widetilde{F}_1(F_2(u_\varepsilon, \dots, u_\lambda), u_{\beta_0}, \dots, u_{\delta_0}). \tag{8}$$

In what follows it is assumed that equation (2) has been taken precisely in this notation (7), i.e. after substituting the values (5) it takes the form

$$u_\alpha = F_2(u_\varepsilon, \dots, u_\lambda). \tag{9}$$

With such a choice of the representation (2), from the equivalence of equation (9) and the equation obtained by substituting in equation (4) the values (5) with $i = \alpha$, it follows that

$$F_2(u_\varepsilon, \dots, u_\lambda) \equiv \theta_\alpha(u_1, \dots, u_{\alpha-1}, u_{\alpha+1}, \dots, u_n)_{\beta=\beta_0,\dots,\delta=\delta_0}.$$

In view of this, equation (2) can be written in the form

$$F_1 = \theta_\alpha(u_1, \dots, u_{\alpha-1}, u_{\alpha+1}, \dots, u_n)_{\beta=\beta_0,\dots,\delta=\delta_0}, \tag{10}$$

where the unknown remains the function F_1.

This function can be determined from any of the identities

$$F_1(u_\alpha, \dots, u_\delta) \equiv$$

$$\equiv \theta_\alpha(u_1, \dots, u_{\alpha-1}, u_{\alpha+1}, \dots, u_n)_{u_\beta=u_{\beta_0},\dots,u_\delta=u_{\delta_0},\ u_p=\theta_p(u_1,\dots,u_{\alpha-1},u_{\alpha+1},\dots,u_n)}, \tag{11}$$

obtained as a result of substituting into equation (10), in place of the variable u_p, $p = \varepsilon, \dots, \lambda$, its expression (4).

After determining the function F_1, the desired representation (2) of equation (1), satisfying condition (9), can be written in final form as

$$\theta_\alpha(u_1, \dots, u_{\alpha-1}, u_{\alpha+1}, \dots, u_n)_{u_\beta=u_{\beta_0},\dots,u_\delta=u_{\delta_0},\ u_p=\theta_p(u_1,\dots,u_{\alpha-1},u_{\alpha+1},u_n)} =$$

$$= \theta_\alpha(u_1, \dots, u_{\alpha-1}, u_{\alpha+1}, \dots, u_n)_{u_\beta=u_{\beta_0},\dots,u_\delta=u_{\delta_0}}. \tag{12}$$

It is obvious that the existence of the identities (11), or, more precisely, the independence of the right-hand sides of these identities from the variables $u_\varepsilon, \dots, u_\lambda$, is a necessary condition for the existence of the representation (2) for equation (1). The sufficiency of any one of them for the existence of this representation follows from the fact that equation (12) determines the same functions (4) as equation (1). This is easily verified by substituting, into the expression for the function $\theta_p = \theta_p(u_1, \dots, u_{p-1}, u_{p+1}, \dots, u_n)$, its expression (4) in equation (12) in place of any one of its arguments. In all such cases equation (12) turns into an identity.

In practice, the separation of variables in a concrete equation with many variables reduces to the following operations. First, one finds solutions of this equation with respect to any two variables u_α and u_p, i.e. one finds the corresponding functions θ_α and θ_p (4).

The next step is to enumerate the admissible substitutions

$$u_\beta = u_{\beta_0}, \dots,\ u_\delta = u_{\delta_0}, \quad u_p = \theta_p(u_1, \dots, u_{p-1}, u_{p+1}, \dots, u_n), \quad p \neq \beta, \dots, \delta,$$

in the expression for the function θ_α up to the case when, as a result, one obtains a function F_1 depending only on $u_\alpha, u_\beta, \dots, u_\delta$.

A case is possible in which the variables u_α and u_p in equation (1) cannot be separated, since both of them are arguments of the function F_1 in the required representation, and therefore the method described leads to a negative result, despite the fact that this representation exists.

To exclude this case, a solution of equation (1) is sought with respect to a third variable $u_q,\ q \neq \beta, \dots, \delta$, and the substitutions

$$u_\beta = u_{\beta_0}, \dots,\ u_\delta = u_{\delta_0}, \quad u_q = \theta_q(u_1, \dots, u_{q-1}, u_{q+1}, \dots, u_n)$$

are tested.

Separation of variables by the method described is carried out very simply in nomographically rational equations (2), i.e. in equations of the form (1) whose left-hand side is a nomographic polynomial (7), containing only one function of each variable.

Example. The nomographically rational equation

$$af_1f_2f_3f_4 + bf_1f_2f_3 + cf_1f_2f_4 + df_1f_3f_4 + ef_2f_3f_4 = 0, \tag{a}$$

where $a, \dots, e$ are arbitrary nonzero coefficients and $f_i = f_i(u_i)$, $i = 1, \dots, 4$, if the f_i are regarded as new variables, has the solution

$$f_1 = -\frac{ef_2f_3f_4}{af_2f_3f_4 + bf_2f_3 + cf_2f_4 + df_3f_4}. \qquad (\ )$$

After the substitution

$$f_2 = f_{20},$$

where f_{20} is an arbitrary nonzero number, equation ( ) yields the expression of the function

$$F_2 = -\frac{e f_{20} f_3 f_4}{f_{20}(a f_3 f_4 + b f_3 + c f_4) + d f_3 f_4}. \qquad (\ )$$

Substitution into ( ), in place of f_3, of the solution of equation (a) with respect to f_3 yields

$$F_1 = -\frac{e f_{20} f_1 f_2}{d f_1 f_2 - f_{20}(d f_1 + e f_2)}. \qquad (\ )$$

Since the function ( ) does not depend on f_3 and f_4, equation (a) admits a representation (2), whose right- and left-hand sides are determined by the equalities ( ) and ( ).

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Note: Figure translations are in progress. See original paper for figures.