ON REGULAR NONUNIFORM GRID SCHEMES AND QUASI-STABILITY

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Abstract

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1. Both for proving the existence and uniqueness of solutions, and for the actual construction of (approximate) solutions of differential equations, the method of grids is often used in the form of *homogeneous* grid schemes. Here homogeneous grid schemes are understood to mean such schemes in which, at all nodes of the given grid, *one and the same* grid equation is used (see [1]). In this paper we shall consider schemes of a different character—*regular nonuniform* grid schemes.

In [2], p. 76, for the numerical integration of the equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$$

an explicit regular nonuniform grid scheme on a square grid was proposed, according to which the next layer is computed in the following two stages ($u_{i,j,k} = u(ih, jh, kl),\ r = l/h^2$):

1st stage. The nodes for which $i + j$, for example, is odd are computed by the usual explicit formula

$$u_{i,j,k+1} = (1 - 4r)u_{i,j,k} + r(u_{i-1,j,k} + u_{i+1,j,k} + u_{i,j-1,k} + u_{i,j+1,k}). \qquad (1)$$

2nd stage. Using these values $u_{i,j,k+1}$, the remaining nodes (for which $i + j$ is even) are also computed explicitly, by the usual "implicit" formula

$$u_{i,j,k+1} = \frac{1}{1+4r}\left[u_{i,j,k} + r(u_{i-1,j,k+1} + u_{i+1,j,k+1} + u_{i,j-1,k+1} + u_{i,j+1,k+1})\right]. \qquad (2)$$

In the cited work it is noted that this "checkerboard" scheme, in comparison with the homogeneous one in which only the classical explicit equation (1) is used (in what follows we shall simply call such a scheme scheme (1)), 1) requires the same number of arithmetic operations to compute one layer, 2) is more accurate (because of the difference in the signs of the temporal components of the errors in equations (1) and (2)), 3) from the point of view of convenience in programming may prove preferable (since the new values $u_{i,j,k+1}$, immediately after their computation, can be sent to the place of the old values $u_{i,j,k}$), 4) in the sense of stability is in any case no worse (since equation (2) is absolutely stable).

Next, for schemes of the type of scheme (1), (2), we introduce the concept of *quasi-stability*, which generalizes (see below) the concept of ordinary stability, and show that scheme (1), (2) is quasi-stable for

$$r \leq \frac{1}{2}. \tag{3}$$

Recall that scheme (1) is stable in the ordinary sense for $r \leq \frac{1}{4}$.

Before proceeding to the derivation of condition (3), we present numerical results for the case of the unit square $0 \leq x, y \leq 1$, the initial function $U(x, y, 0) = 16xy(1-x)(1-y)$, and zero boundary conditions. For $h = 1/23,\ y = 10/23,\ t = 50/529$ we have (see Table 1):

Table 1

x	Exact solution	Numerical solution, obtained by the scheme	Numerical solution, obtained by the scheme	Numerical solution, obtained by the scheme	Numerical solution, obtained by the scheme
x	Exact solution	(1), $r = 1/4$	error	(1), (2), $r = 1/2$	error
0	0,000000	0,000000	0,000000	0,000000	0,000000
$2h$	043552	043298	254	043598	–046
$4h$	083872	083384	488	084039	–167
$6h$	117971	117284	687	118295	–324
$8h$	143318	142485	833	143754	–436
$10h$	158037	157118	919	158514	–477

It follows from this table (at least for the given example) that 1) with twice as few arithmetic operations, the "checkerboard" scheme (1), (2), in comparison with scheme (1), has approximately twice as small an error (in the three-dimensional case, as numerical calculations show, the increase in accuracy of the "checkerboard" scheme in comparison with the usual explicit scheme, with twice as few arithmetic operations, is still more substantial), and 2) condition (3) is sufficient (and, as numerical calculations for $r > 1/2$ show, necessary) for stability in the ordinary sense of the "checkerboard" scheme (1), (2).

2. To investigate the quasi-stability of scheme (1), (2), we rewrite formula (2), with the aid of formula (1), in a form explicitly solved with respect to $u_{i,j,k+1}$. As a result we obtain

$$u_{i,j,k+1} = \frac{1}{1+4r}\Big[(1+4r^2)u_{i,j,k}+r(1-4r)(u_{i-1,j,k}+u_{i+1,j,k}+u_{i,j-1,k}+u_{i,j+1,k})+2r^2(u_{i+1,j+1,k}+u_{i-1,j+1,k}+u_{i+1,j-} \tag{4}$$

Putting $u_{i,j,k+1} = Au_{i,j,k}$, where A is a linear operator determined by the grid equations (1) and (4), we introduce auxiliary functions of Neumann type

$$w_{1;i,j} = e^{\sqrt{-1}(\alpha i+\beta j)}, \qquad w_{2;i,j} = (-1)^{i+j}e^{\sqrt{-1}(\alpha i+\beta j)} \tag{5}$$

(α and β are arbitrary real numbers). With the aid of these auxiliary "basis" functions we define the operator A in the following way:

$$Aw_{1;i,j} = a_{11}w_{1;i,j} + a_{21}w_{2;i,j}, \tag{6}$$

$$Aw_{2;i,j} = a_{12}w_{1;i,j} + a_{22}w_{2;i,j}, \tag{7}$$

i.e., we associate with the operator A the matrix

$$\widetilde{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{8}$$

Definition. The scheme (1), (2) (or, equivalently, the scheme (1), (4)) will be called quasi-stable if and only if the roots of the secular equation

$$\det|\widetilde{A} - \lambda E| \equiv \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0 \tag{9}$$

(E is the identity matrix) do not exceed 1 in modulus.

The above choice of the functions (5), satisfying the relations

$$w_{1;i,j} = -w_{2;i,j} \quad (i+j \text{ odd}),$$

$$w_{1;i,j} = w_{2;i,j} \quad (i+j \text{ even}), \tag{10}$$

makes it possible to write, in accordance with (1) and (4), the quantities $Aw_{1;i,j}$ and $Aw_{2;i,j}$ in the following form (we omit the elementary calculations here):

$$Aw_{1;i,j} = \begin{cases} (1-4r+2r\gamma)\, w_{1;i,j}, & (i+j \text{ odd}), \\ \dfrac{1}{1+4r}\left[1+4r^2\gamma^2+2r\gamma(1-4r)\right] w_{1;i,j}, & (i+j \text{ even}), \end{cases} \tag{11}$$

$$Aw_{2;i,j} = \begin{cases} (-1 + 4r + 2r\gamma)\, w_{1;i,j}, & (i+j \text{ odd}), \\ \dfrac{1}{1+4r}\,[1 + 4r^2\gamma^2 - 2r\gamma(1-4r)]\, w_{1;i,j}, & (i+j \text{ even}), \end{cases} \tag{12}$$

where, for brevity of notation, we have set $\gamma = \cos\alpha + \cos\beta$.

Using again the equalities (10), from (6) and (11) we directly obtain a system of two linear algebraic equations

$$a_{11} - a_{21} = 1 - 4r + 2r\gamma, \qquad a_{11} + a_{21} = \frac{1}{1+4r}\,[1 + 4r^2\gamma^2 + 2r\gamma(1-4r)],$$

whose solution has the form

$$a_{11} = \frac{1}{1+4r}\,(1 + 2r\gamma - 8r^2 + 2r^2\gamma^2), \qquad a_{21} = \frac{2r^2(\gamma-2)^2}{1+4r}. \tag{13}$$

Similarly, from (7) and (12) we have

$$a_{12} = \frac{2r^2(\gamma+2)^2}{1+4r}, \qquad a_{22} = \frac{1}{1+4r}\,(1 - 2r\gamma - 8r^2 + 2r^2\gamma^2). \tag{14}$$

Formulas (13)–(14) define the matrix $\widetilde{A} = \widetilde{A}(\alpha, \beta, r)$ of second order. In order that the roots of its characteristic quadratic equation (9) not exceed 1 in modulus, it is known to be necessary and sufficient that the inequalities

$$|a_{11} + a_{22}| \leqslant 1 + a_{11}a_{22} - a_{12}a_{21} \leqslant 2,$$

be satisfied; by virtue of (13), (14), these can be written in the form

$$\frac{2}{1+4r}\,|1 - 8r^2 + 2r^2\gamma^2| \leqslant 1 + \frac{1-4r}{1+4r} \leqslant 2.$$

But it is easy to verify directly that these inequalities are satisfied for every $|\gamma| \leqslant 2$ and every $0 \leqslant r \leqslant 1/2$. Thus, the quasi-stability condition (3) for the scheme (1), (2) has been established.

3. If, for example, the even layers are computed by the "checkerboard" scheme (1), (2), and the odd layers by the analogous "checkerboard" scheme shifted by a step h, then we obtain a regular nonuniform scheme having the following property: if at some node (i, j, k) formula (1) (respectively (2)) is used, then at the six neighboring nodes $(i-1, j, k)$, $(i+1, j, k)$, $(i, j-1, k)$, $(i, j+1, k)$, $(i, j, k-1)$, $(i, j, k+1)$ formula (2) (respectively (1)) is used;

that is, the classical explicit and implicit difference equations are interlaced not only in the directions of the spatial coordinates x, y, but also in the direction of the time coordinate t. This scheme is, evidently, explicit; for the computation of one layer it requires the same number of arithmetic operations as scheme (1), and, which is an interesting fact, is absolutely quasi-stable.

However, it should be noted that such a "checkerboard"scheme in all the variables x, y, t, while being absolutely quasi-stable, requires for convergence that the condition $l = o(h)$ be satisfied. Otherwise the grid characteristic "cone," whose "generators" make with the plane xoy the angle

$$\left|\operatorname{arc\,tg}\frac{l}{h}\right|,$$

will not, as $h \to 0$ and $l \to 0$, tend to the characteristic plane $t = \text{const}$ of the given differential equation.

4. Similarly one can show that the semi-explicit regular nonuniform difference scheme (see [2], p. 75)

$$\frac{1+2r}{r}u_{i,j,k+1} - (u_{i-1,j,k+1} + u_{i+1,j,k+1}) = \frac{1-2r}{r}u_{i,j,k} + u_{i,j-1,k} + u_{i,j+1,k} \quad (j \text{ even}), \tag{15}$$

$$u_{i,j,k+1} = \frac{1-2r}{1+2r}u_{i,j,k} + \frac{r}{1+2r}(u_{i-1,j,k} + u_{i+1,j,k} + u_{i,j-1,k+1} + u_{i,j+1,k+1}) \quad (j \text{ odd}) \tag{16}$$

is quasi-stable for $r \leqslant 1$. Recall that the homogeneous scheme in which only equation (15) is used (below we shall call such a scheme simply scheme (15)) is stable in the usual sense for $r \leqslant 1/2$. In terms of accuracy and programming convenience, scheme (15), (16), as compared with scheme (15), is analogous to scheme (1), (2) as compared with scheme (1); moreover, scheme (15), (16), as compared with scheme (15), requires ≈ 1.37 times fewer arithmetic operations for computing one layer (since only half the nodes are computed by means of sweep formulas).

If the odd layers are computed by formulas (15), (16), for example, and the even layers by analogous formulas in which only the "sweep" is carried out in the direction of the other coordinate, then we obtain a modified method of alternating directions. We give numerical results for the case of the unit square with zero boundary conditions and initial function $U(x, y, 0) = 10^4 \sin \pi x \sin \pi y$. For $h = 0.05$, $t = 0.5$, and $x = 0.4$ we have (see Table 2).

Here, in the last three columns, the numerical solutions are given, obtained respectively by means of scheme (1), the ordinary alternating-direction method (only equation (15) is used, and the analogous one for the other coordinate), and the modified alternating-direction method. It follows from Table 2 that the first method, with a 16-fold smaller

Table 2

y	Exact solution	$r = \frac{1}{4}$	$r = 4$	$r = 4$
0,00	0,000000	0,000000	0,000000	0,000000
10	152011	145925	150233	150275
20	289141	277566	285721	285760
30	397969	382037	393317	393415
40	467840	449112	462371	462384
50	491917	472224	486166	486273

in the time step, in comparison with the other two methods, is considerably less accurate. The modified alternating-direction method, with approximately 1.37 times fewer arithmetic operations, is somewhat more accurate than the usual alternating-direction method.

5. More generally, the question of regular nonuniform grid schemes may be considered as follows. Let the set of all nodes of a regular (square, triangular, etc.) grid be divided into s nonintersecting subsets $M_1, M_2, \dots, M_s$ in some regular (cyclic) manner, and let us use, at the nodes belonging to M_p ($p = 1, 2, \dots, s$), the grid equation $L_{hp}u = f$, approximating the given differential equation $LU = f$. A scheme in which the sets $M_1, M_2, \dots, M_s$ are computed (for a given t) in a definite order will be called a regular nonuniform grid scheme of rank s. In this sense, all uniform grid schemes are regular nonuniform schemes of rank 1. The schemes considered above, (1), (2) and (15), (16), are regular nonuniform schemes of rank 2.

In the general case of regular nonuniform grid schemes of rank s, the investigation of quasi-stability reduces to the investigation of the eigenvalues of a matrix (or matrices, in the case of semi-explicit regular nonuniform grid schemes) of order s (for $s = 2$, for example, (8)). Here, for $s = 1$, this investigation of quasi-stability is expressed in the well-known Neumann method for investigating stability. In this sense, the quasi-stability defined above is a generalization, for the case $s > 1$, of ordinary stability.

The concept of quasi-stability introduced above is also a generalization of the ordinary concept of stability in the following sense. Suppose that the grid operators L_{hp} depend on certain parameters and that, for certain values of these parameters, the regular nonuniform grid scheme passes into the corresponding uniform grid scheme (i.e. $L_{hp} = L_h$, $p = 1, 2, \dots, s$). Then the condition of quasi-stability for the regular nonuniform grid scheme passes into the condition of stability (in the ordinary sense) of the corresponding uniform grid scheme.

On the basis of the regular nonuniform grid schemes considered above, one can automatically obtain a number of iterative methods for solving systems of elliptic grid equations. As numerical computations have shown, the iterative method based on the modified alternating-direction method described above has proved to be very effective.

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Note: Figure translations are in progress. See original paper for figures.