ON ISOINVOLUTIVE DECOMPOSITIONS OF LIE ALGEBRAS

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ON ISOINVOLUTIVE DECOMPOSITIONS OF LIE ALGEBRAS

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In the present paper, for compact real semisimple Lie algebras, the theory of isoinvolutive decompositions is set forth (without proofs) and its application to compact symmetric spaces of rank 1.

Let Γ_r be a compact semisimple Lie algebra over the field of real numbers; let $e_1, e_2, \dots, e_r$ be its basis. The structure is then determined by the commutators of the basis vectors

$$[e_I e_K] = c^R_{IK} e_R, \qquad c^R_{IK} = -c^R_{KI}, \qquad I, K, R = 1, \dots, r, \tag{1}$$

and by the Jacobi identities

$$[e_I[e_{Je}K]] + [e_K[e_{Ie}J]] + [e_J[e_{Ke}I]] = 0. \tag{2}$$

The transformations of the adjoint group Γ^*_r have the form

$$\bar{\eta} \to e^{\bar{c}(u)}\bar{\eta}, \qquad \bar{c}(u) = \|c^I_{JK} u^K\|, \tag{3}$$

and the tensor c^R_{IK} is invariant under such transformations.

In a compact semisimple Lie algebra one can introduce a positive definite metric tensor

$$a_{IJ} = c^P_{IK} c^K_{PJ}, \qquad a_{IJ} = a_{JI} \tag{4}$$

and consider orthonormal bases, in which

$$a_{IJ} = \delta_{IJ}. \tag{5}$$

Then, as is known,

$$c^I_{JK} = -c^J_{IK}. \tag{6}$$

An automorphism A of the algebra Γ is called an **involomorphism** if

$$A^2 = I, \qquad A \neq I. \tag{7}$$

In some orthobasis any involomorphism S has a diagonal form with ± 1 on the main diagonal.

All $\eta \in \Gamma_r$ such that $S\eta = \eta$ form a subalgebra (invoalgebra) $L \subset \Gamma_r$; the pair Γ_r/L will be called an **invopair**. Obviously,

$$\Gamma_r = E + L, \qquad E \perp L; \qquad S\bar{\eta} = \begin{cases} \bar{\eta}, & \bar{\eta} \in L, \\ -\bar{\eta}, & \bar{\eta} \in E. \end{cases} \tag{8}$$

The invopair Γ/L generates the symmetric space Γ^*/L^*, where L^* has on E an irreducible linear representation, if Γ is simple.

Definition 1. We shall say that the algebra Γ is an **involutive sum of its subalgebras** $\Gamma = L_1 + L_2 + L_3$, if L_1, L_2, L_3 are the invoalgebras of commuting involomorphisms $S_1, S_2, S_3 = S_1 S_2$, respectively.

Obviously, $L_1 \cap L_2 = L_1 \cap L_3 = L_2 \cap L_3 = L_0$, and L_i/L_0 $(i = 1, 2, 3)$ are invopairs.

Definition 2. An involutive sum $\Gamma = L_1 + L_2 + L_3$ will be called **isoinvolutive** if L_1 and L_2 are conjugate in Γ^*, with the conjugating automorphism of the form $e^{\bar{c}(\bar{\xi}t)}$, where $\bar{\xi} \in E_3 = L_3 - L_0$.

Lemma 1. *If Γ/L is a compact semisimple involpara, then there exists $\bar{\xi} \in E = \Gamma - L$ such that the subgroup $e^{c(\bar{\xi}t)}$ in Γ^* is compact.*

Corollary 1. *If Γ is a semisimple compact Lie algebra and S is its involmorphism, then there exist inner automorphisms Q_n in Γ^* (n a positive integer) such that $(Q_n)^n = 1$, $SQ_{nS} = Q_n^{-1}$.*

On the basis of Lemma 1 and its corollary the following theorem is proved:

Theorem 1. *Let Γ be a semisimple compact Lie algebra, and let L_1 be any of its invoalgebras; then there exists an isoinvolutive decomposition*

$$\Gamma = L_1 + L_2 + L_3, \qquad L_1 = \varphi L_2, \quad \varphi \in \Gamma^*, \quad \varphi^4 = 1, \quad S_3 = \varphi^2.$$

Corollary 1. *The restriction of φ to L_3 acts as an involmorphism or as the identity automorphism.*

Corollary 2. *The restriction of φ to L_0 acts as an involmorphism or the identity automorphism.*

Corollary 3. *If the restriction of φ to L_3 acts as an involmorphism, then L_3 is an isoinvolutive sum $L_3 = L_0 + L_0' + L_0''$, where L_0, L_0', L_0'' are the invoalgebras of the involmorphisms S_1, $(\varphi^{1/2})^{-1}S_1(\varphi^{1/2})$, φ, respectively.*

Definition 3. An isoinvolutive sum will be called **of type I** if the restriction of the conjugating automorphism φ to L_3 is the identity automorphism.

We shall consider only simple compact algebras Γ.

Let $\Pi = L_0 \cap L_0'$; then

$$L_0 = \Pi + E_0, \qquad L_0' = \Pi + E_0', \qquad L_0'' = \Pi + E_0'',$$

$$L_1 = L_0 + E_1, \qquad L_2 = L_0 + E_2,$$

where E, E', E'' are orthogonal to Π, and E_1, E_2 are orthogonal to L_0. From Theorem 1 and its corollaries we have

$$E_2 = \varphi E_1, \qquad E_0' = (\varphi^{1/2})E_0, \qquad (\varphi^{1/2})\Pi = \Pi, \qquad (\varphi^{1/2})E_0'' = E_0''. \tag{9}$$

Choosing orthonormal bases in Π, E_1, E_0, E_0'', with the help of (9) we construct bases in E_2 and E_0'. The union of all the indicated bases generates an orthobasis in Γ, which we shall call **isoinvolutive**. Let, in this notation,

$$\begin{array}{lll} X_{i_1} & (i = 1, \dots, n) & \text{be a basis in } E_1, \\ X_{i_2} & (i = 1, \dots, n) & \text{be a basis in } E_2; \\ Y_{\alpha_2} & (\alpha = 1, \dots, \rho) & \text{be a basis in } E_0, \\ Y_{\alpha_1} & (\alpha = 1, \dots, r) & \text{be a basis in } \Pi; \end{array} \left.\begin{array}{l} \\ \\ \end{array}\right\} Y_\alpha \tag{10}$$

$$\left.\begin{array}{lll} V_{\alpha_2} & (\alpha = 1, \dots, \rho) & \text{be a basis in } E_0', \\ V_{\alpha_1} & (a = 1, \dots, m) & \text{be a basis in } E_0''. \end{array}\right\} V_\alpha$$

By construction

$$\begin{gathered} \varphi X_{i_1} = X_{i_2}, \qquad \varphi X_{i_2} = -X_{i_1}, \qquad \varphi Y_\alpha = \varphi_\alpha^\beta Y_\beta \\ \left(\varphi Y_{\alpha_2} = -Y_{\alpha_2}, \qquad \varphi Y_{\alpha_1} = Y_{\alpha_1}\right), \\ \varphi V_\alpha = \varphi_\alpha^b V_b \qquad \left(\varphi V_{\alpha_2} = -V_{\alpha_2}, \qquad \varphi V_{\alpha_1} = V_{\alpha_1}\right). \end{gathered} \tag{11}$$

The automorphism $\theta = \varphi^{1/2}$ acts, as is easily verified, as follows:

$$\theta X_{i_1} = \frac{1}{\sqrt{2}}\theta_i^j(X_{j_1} + X_{j_2}), \quad \theta X_{i_2} = \frac{1}{\sqrt{2}}\theta_i^j(-X_{j_1} + X_{j_2}),$$

$$\theta_k^p \theta_s^k = \delta_s^p, \qquad \theta_j^i = \theta_i^j,$$

$$\theta Y_{a_2} = V_{a_2}, \quad \theta V_{a_2} = -Y_{a_2}, \quad \theta Y_{a_1} = \theta^{\beta_1}_{a_1} Y_{\beta_1}, \quad \theta V_{a_1} = \theta^{c_1}_{a_1} V_{c_1},$$

$$\theta^{\lambda_1}_{a_1}\theta^{a_1}_{\mu_1} = \delta^{\lambda_1}_{\mu_1}, \quad \theta^{c_1}_{a_1}\theta^{a_1}_{b_1} = \delta^{c_1}_{b_1}, \quad \theta^{\beta_1}_{\alpha_1} = \theta^{\alpha_1}_{\beta_1}, \quad \theta^{c_1}_{a_1} = \theta^{a_1}_{c_1}. \tag{12}$$

Taking into account the action of the automorphisms S_1, S_2, S_3, we write the structure of the algebra Γ in the form

$$[X_{i_1} X_{j_1}] = -b^a_{i_1 j_1} Y_a, \quad [X_{i_2} X_{j_2}] = -b^a_{i_2 j_2} Y_a, \quad [X_{i_1} X_{j_2}] = q^a_{i_1 j_2} V_a,$$

$$[X_{i_1} V_a] = t^{j_2}_{i_1 a} X_{j_2}, \quad [X_{p_2} V_a] = t^{j_1}_{p_2 a} X_{j_1}, \quad [V_{aV} c] = -b^{\alpha}_{ac} Y_{\alpha}, \quad [X_{i_1} Y_{\alpha}] = a^{p_1}_{i_1 \alpha} X_{p_1}, \tag{13}$$

$$[X_{i_2} Y_{\alpha}] = a^{p_2}_{i_2 \alpha} X_{p_2}, \quad [V_{cY_\alpha}] = a^b_{c\alpha} V_b, \quad [Y_\alpha Y_\beta] = c^{\gamma}_{\alpha\beta} Y_\gamma.$$

Introduce the notation

$$a^{p_1}_{k_1 \alpha} = a^p_{k\alpha}, \quad b^a_{k_1 j_1} = b^a_{kj}, \quad q^a_{k_2 j_1} = q^a_{kj}, \quad t^{p_2}_{i_1 a} = t^p_{ia}. \tag{14}$$

Then the automorphism φ from (11), for the structure (13), gives

$$b^{\alpha}_{i_2 j_2} = \varphi^{\alpha}_{\beta} b^{\beta}_{ij}, \quad q^b_{ij} = \varphi^b_a q^a_{ji}, \quad t^{j_1}_{i_2 b} = -\varphi^a_b t^j_{ia}, \quad a^{p_2}_{i_2 \alpha} = \varphi^{\beta}_{\alpha} a^p_{i\beta},$$

$$b^{\alpha}_{ac} = \varphi^f_a \varphi^d_c \varphi^{\alpha}_{\beta} b^{\beta}_{fd}, \quad a^b_{c\alpha} = \varphi^d_c \varphi^b_f \varphi^{\beta}_{\alpha} a^f_{d\beta}, \quad c^{\gamma}_{\alpha\beta} = \varphi^{\lambda}_{\alpha}\varphi^{\mu}_{\beta}\varphi^{\gamma}_{\nu} c^{\nu}_{\lambda\mu}. \tag{15}$$

From the orthonormality of the isoinvobasis it also follows that

$$a^i_{j\alpha} = b^{\alpha}_{ji}, \quad q^a_{kj} = t^k_{ja}, \quad t^{j_1}_{i_2 b} = -t^i_{jb}, \quad a^f_{c\alpha} = b^{\alpha}_{cf}, \quad c^{\alpha}_{\beta\gamma} = -c^{\alpha}_{\beta\gamma}. \tag{16}$$

The automorphism θ from (12) for the structure (13) leads to the relations

$$t^p_{i\alpha_2} = \theta^p_i \theta^k_i a^s_{k\alpha_2}, \quad \theta^k_i \theta^b_s t^p_{ia} = \theta^b_a t^s_{kb}, \quad \theta^k_i \theta^p_s a^s_{k\beta_1} = \theta^{\alpha_1}_{\beta_1} a^p_{i\alpha_1},$$

$$c^{\lambda_1}_{\alpha_2\beta_2} = -b^{\sigma_2}_{\alpha_2\beta_2}\theta^{\lambda_1}_{\sigma_1}, \quad b^{\sigma_2}_{a_1\beta_2} = \theta^{c_1}_{a_1} a^{\sigma_2}_{c_1\beta_2}, \quad a^{e_1}_{\alpha_2 c_2} = \theta^{e_1}_{b_1} a^{b_1}_{c_2\alpha_2}, \quad a^{b_2}_{e_2\alpha_1} = \theta^{\beta_1}_{\alpha_1} c^{b_2}_{e_2\beta_1}, \tag{17}$$

$$a^{b_1}_{c_1\alpha_1}\theta^{e_1}_{b_1} = \theta^{d_1}_{c_1}\theta^{\beta_1}_{\alpha_1} a^{e_1}_{d_1\beta_1}, \quad c^{\lambda_1}_{\alpha_1\beta_1}\theta^{\mu_1}_{\lambda_1} = \theta^{\sigma_1}_{\alpha_1}\theta^{\nu_1}_{\beta_1} c^{\mu_1}_{\sigma_1\nu_1}.$$

Finally, consideration of all Jacobi identities leads to the relations

$$a^p_{k\alpha} b^\alpha_{ij} = 2a^p_{k\alpha_2} b^{\alpha_2}_{ij} + t^p_{j\alpha_2} q^{\alpha_2}_{ki} - t^p_{i\alpha_2} q^{\alpha_2}_{kj} + t^p_{j\alpha_1} q^{\alpha_1}_{ki} - t^p_{i\alpha_1} q^{\alpha_1}_{kj},$$

$$-b^{b_1}_{\alpha_1 c_1} a^p_{i\alpha_1} = t^p_{ja} t^j_{ic_1} - t^p_{jc_1} t^j_{ia_1},$$

$$-a^{b_1}_{c_2\alpha_2} t^k_{ib_1} = t^k_{pc_2} a^p_{i\alpha_2} + a^k_{p\alpha_2} t^p_{ic_2},$$

$$a^{b_2}_{c_1\alpha_2} t^k_{ib_2} = t^k_{pc_1} a^p_{i\alpha_2} + a^k_{p\alpha_2} t^p_{ic_1},$$

$$a^{b_1}_{c_1\alpha_1} t^k_{ib_1} = a^k_{p\alpha_1} t^p_{ic_1} - t^k_{pc_1} a^p_{i\alpha_1}, \qquad (18)$$

$$c^{\sigma_1}_{\alpha_2\beta_2} a^l_{j\sigma_1} = a^l_{i\beta_2} a^i_{j\alpha_2} - a^l_{i\alpha_2} a^i_{j\beta_2},$$

$$c^{\sigma_2}_{\beta_2\alpha_1} a^l_{j\sigma_2} = a^l_{i\beta_2} a^i_{j\alpha_1} - a^l_{i\alpha_1} a^i_{j\beta_2},$$

$$c^{\sigma_1}_{\alpha_1\beta_1} a^l_{j\sigma_1} = a^l_{i\beta_1} a^i_{j\alpha_1} - a^l_{i\alpha_1} a^i_{j\beta_1}.$$

Definition 4. An isoinvolutive sum, if it is not of type I, shall be called **type II** if $\theta^p_s = +\delta^p_s$, and **type III** otherwise.

The use of the relations obtained above leads to the theorem:

Theorem 2. If a compact simple Lie algebra Γ admits an isoinvolutive decomposition

$$\Gamma = L_1 + L_2 + L_3$$

of type I or II, then the maximal subalgebra of elements fixed under the action of the conjugating auto-

morphism φ, has center $\xi \in L_3 - L_0$. Hence it follows that, if the involara Γ/L_1 has rank 1, then $L_3 - L_0$ is one-dimensional for type I, and $L_0'' - \Pi$ is one-dimensional for type II.

This permits one to obtain the following theorems:

Theorem 3. If the isoinvolutive sum of type I

$$\Gamma = L_1 + L_2 + L_3$$

is simple and compact, and Γ/L_1 is an involara I, then Γ/L_1 is of the form

$$SO(n+1)/SO(n),$$

where L_1/L_0 is of the form

$$SO(n)/SO(n-1)$$

(with the natural embedding).

Theorem 4. If Γ/L_1 is a simple compact involara of rank 1, and

$$\Gamma = L_1 + L_2 + L_3$$

is an isoinvolutive decomposition of type II and E_0 is one-dimensional, then Γ/L_1 is of the form

$$SU(n+1)/S(U(n) \times \tilde{U}(1)),$$

and $L_1/\tilde{L}_0$ is of the form

$$SU(n)/S(U(n-1) \times U(1))$$

(with the natural embeddings)

$$(L_0 - \tilde{L}_0 = Z$$

is a one-dimensional center in L_1).

Theorem 5. If Γ/L_1 is a simple compact involara of rank 1,

$$\Gamma = L_1 + L_2 + L_3$$

is an isoinvolutive decomposition of type II and E_0 is three-dimensional, then Γ/L_1 is of the form

$$Sp(n+1)/Sp(n) \times Sp(1),$$

and $L_1/\tilde{L}_0$ is of the form

$$Sp(n)/Sp(n-1) \times Sp(1)$$

(with the natural embeddings)

$$(L_0 - \tilde{L}_0 = Z$$

is a three-dimensional ideal in L_1).

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Note: Figure translations are in progress. See original paper for figures.