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Abstract

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ON THE ORDER OF CONVERGENCE OF CUBATURE FORMULAS

The subject of the present note is an estimate of the norm of the error of cubature formulas in a domain Ω of n independent variables.

We shall consider the error of a cubature formula as a linear functional of the form:

$$l(x) = \mathcal{E}_\Omega(x) - \sum_{k=1}^{N} c_k \delta(x - x^{(k)}) \tag{1}$$

in the space $L_2^{(m)}(E_n)$ of classes of functions possessing derivatives of order m, with integrable square and norm

$$\|\varphi\|_{L_2^{(m)}} = \left\{ \int_{E_n} \sum_{|\alpha|=m} (D^\alpha \varphi)^2 \, dx \right\}^{1/2}. \tag{2}$$

$L_2^{(m)}(E_n)$ is the factor space of $W_2^{(m)}$ by the space of polynomials of degree $m-1$. By $\mathcal{E}_\Omega(x)$ is denoted the characteristic function of the domain Ω; the symbol $|\alpha|$ for an integral vector α, as usual, means $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}$.

It is necessary to assume here that

$$m > n/2; \tag{3}$$

$$(l(x), x^\alpha) = 0 \quad \text{for } |\alpha| < m. \tag{4}$$

Let $|\Omega|$ be the volume of the domain Ω. Put

$$|\Omega|/N = h^n. \tag{5}$$

Theorem 1. *There exists a constant K_1, depending only on the numbers m, n, such that*

$$\|l\|_{L_2^{(m)*}} \geq K_1 \sqrt{|\Omega|}\, h^m. \tag{6}$$

The proof of this theorem is based on estimates analogous to those given by N. S. Bakhvalov for the proof of theorems of a similar kind in other function spaces.

Let us introduce a definition. We shall say that a certain cube $Q_j(|x_s - x_s^{(j,0)}| < k/2)$ with edge length k, containing no points $x^{(t)}$ on its boundary or near it at a distance ηk, where $\eta > 0$, is a cube with an insufficient specification of the functional $l(x)$, if

$$(l(x), \mathcal{E}_{Q_j}(x)) > \eta_2 k^n, \tag{7}$$

where $\eta_2 > 0$.

Lemma 1. *If in the domain Ω one can indicate a nonoverlapping system of cubes Q_j $(j = 1, 2, \dots, N_1)$ with insufficient specification of the functional $l(x)$ and with sum of volumes greater than a certain constant*

$$\sum_{j=1}^{N_1} |Q_j| > \Omega_1,$$

then the norm of the functional $l(x)$ satisfies the inequality

$$\|l(x)\|_{L_2^{(m)*}} > K\sqrt{|\Omega_1|}\, k^m. \tag{8}$$

The proof consists, as in N. S. Bakhvalov, in directly estimating the value of the functional $l(x)$ on a certain function consisting of a sum of "caps" over each cube with insufficient specification of $l(x)$.

Lemma 1 implies our theorem.

Partition the domain Ω into cubes by means of a cubic lattice with side $k_1 = 2^{-1/n}h$, and consider all cubes with side $k < (1-\eta_3)k_1$ concentric with the cubes of this lattice. Their number, obviously, will be no less than $2N$. Since they have no common points, at least half of them will contain no $x^{(t)}$, and, consequently, they will be cubes with insufficient specification of the functional. Their total volume $|\Omega_1|$ will be $|\Omega_1| > (1-\eta_3)^n |\Omega|/2$, whence the theorem follows.

Lemma 1 also shows the main source of the error of a cubature formula, namely the nonuniformity of the distribution of the points $x^{(t)}$, which cannot be made ideal.

The estimate given by Theorem 1 is attainable, as follows from the following theorem.

Theorem 2. *Let the error functional $l(x)$ be representable in the form*

$$l(x) = \sum_{\gamma} l_{\gamma}(x/h - \gamma), \tag{9}$$

where γ runs over all points of the integer lattice, and $l_{\gamma}(y)$ satisfies the conditions

$$(l_{\gamma}(y), y^{\alpha}) = 0, \ \ |\alpha| \leq m; \tag{10}$$

$$\|l_{\gamma}(y)\|_{L_2^{(m)*}} < A; \tag{11}$$

$$S\{l(y)\} \in \mathcal{E}(|y| < L) \tag{12}$$

(by $S\{l(y)\}$ is denoted the support of $l(y)$).

Then for the norm of $l(x)$ the inequality is valid

$$\|l(x)\|_{L_2^{(m)*}} < K_2 h^m, \tag{13}$$

where the constant K_2 depends on the shape of the domain Ω and on the numbers A and Γ, but does not depend on the form of the functionals $l_{\gamma}(y)$.

Before proving this theorem, it is useful to note that for domains with piecewise smooth boundary, with the numbers A and L chosen once and for all for all such domains, and for sufficiently small h, one can always construct an infinite set of functionals admitting the representation (9).

Indeed, we can always decompose, for example, the domain Ω into the sum

$$\Omega = \bigcup \Omega_{\gamma}, \tag{14}$$

where Ω_{γ} is a cell lying at a distance not greater than Lh from the point $x = h\gamma$:

$$d(\Omega_{\gamma}, h\gamma) < Lh. \tag{15}$$

The characteristic function $\mathcal{E}_{\Omega_{\gamma}}(x)$ is written in the form

$$\mathcal{E}_{\Omega_{\gamma}}(x) = \mathcal{E}_{\Omega_{\gamma}^{*}}(x/h - \gamma), \tag{16}$$

where $\mathcal{E}_{\Omega^*_\gamma}$ is the characteristic function of some domain with finite support.

For the domain Ω^*_γ, the classical extrapolation method makes it possible to construct a cubature formula with nodes at lattice points such that

$$h(\gamma + \gamma') \in \Omega \tag{17}$$

and $|\gamma'| < L$, exact for all polynomials of degree $m-1$, with error functional

$$l_\gamma(y) = \mathcal{E}_{\Omega^*_\gamma}(x) - \sum C_\gamma^{(\gamma')} \delta(y - \gamma'). \tag{18}$$

The functional $l(x)$, constructed by formula (9), will, as can be verified, satisfy all the conditions of the theorem.

Let us indicate the idea of the proof of Theorem 2. As was established earlier (see (1–3)), the norm of the functional $l(x)$ is expressed in terms of the solution of the polyharmonic equation

$$\Delta^m u = (-1)^{m+1} l(x) \tag{19}$$

in the form

$$\|l\|_{L_2^{(m)*}} = |(l,u)|/\|u\|_{L_2^{(m)}} = \|u\|_{L_2^{(m)}} \tag{20}$$

by means of the elementary solution of equation (19)

$$G(x) = \varkappa r^{2m-n} \begin{cases} \ln r & \text{for even } n, \\ 1 & \text{for odd } n. \end{cases} \tag{21}$$

Here it is convenient to use the known representation of the scalar product:

$$(\varphi, \psi) = (\varphi(x) * \psi(-x))\big|_{x=0}. \tag{22}$$

From formula (20) we obtain

$$\|l\|^2_{L_2^{(m)*}} = |(l,u)| = (l(x) * G(x) * l(-x)). \tag{23}$$

Since the functions $l(x)$ and $l(-x)$ are finite, the triple convolution on the right-hand side of (23) is associative and commutative. Substituting into (22) the expressions for $l(x)$ and $l(-x)$ from (9), we shall have

$$\|l\|^2_{L_2^{(m)}} \leq \sum_{\gamma_1}\sum_{\gamma_2}\left|l_{\gamma_1}\left(\frac{x}{h}-\gamma_1\right) * G(x) * l_{\gamma_2}\left(-\frac{x}{h}-\gamma_2\right)\right|_{x=0} =$$

$$= \sum_{\gamma_1}\sum_{\gamma_2}\left[G(x) * \left(l_{\gamma_1}\left(\frac{x}{h}\right) * l_{\gamma_2}\left(\frac{-x}{h}\right)\right)\right]_{x=h(\gamma_1+\gamma_2)}. \tag{24}$$

It is not difficult to establish the formulas

$$l_1\left(\frac{x}{h}\right) * l_2\left(\frac{x}{h}\right) = h^n l_3\left(\frac{x}{h}\right), \qquad \text{where } l_3(y) = l_1(y) * l_2(y), \tag{25}$$

and also

$$\left\|l\left(\frac{x}{h}\right)\right\|_{L_2^{(m)*}} = h^{n/2+m}\|l(y)\|_{L_2^{(m)*}}; \tag{26}$$

$$\|\varphi(x)\|_{L_2^{(m)}(hx\in\Omega)} = h^{n/2-m}\|\psi(y)\|_{L_2^{(m)}(y\in\Omega)}, \tag{27}$$

where $\psi(y) = \varphi(hy)$, i.e. $\psi(x) = \varphi(x/h)$.

The functional $l_3(y)$ has the properties

$$\|l_3(y)\|_{L_2^{(2m)*}} \leq A^2; \tag{28}$$

$$(l_3(y), y^\alpha) = 0, \qquad |\alpha| \leq 2m+1; \tag{29}$$

$$S\{l_3(y)\} \subset \mathcal{E}(|y| < 2L). \tag{30}$$

Lemma 2. The estimate is valid

$$G(x) * l_{\gamma_1}\left(\frac{x}{h}\right) * l_{\gamma_2}\left(-\frac{x}{h}\right) \leq K\frac{h^{2m+2n+2}}{[h^2+|x|^2]^{n/2+1}}. \tag{31}$$

Inequality (31) for $|x| < 3Lh$ is established directly from (25)–(30). To obtain (31) for $|x| > 3Lh$, we expand $G(x-y)$ in a series in powers of y about the point $y = 0$. We obtain:

$$G(x-y) = \sum_{|\alpha|<2m+2}\frac{(-y)^\alpha}{\alpha!}D^\alpha G(x) + R_{2m+2}(x,y), \tag{32}$$

where, evidently, R_{2m+2} in the domain $|y| < 2L$ satisfies the inequality

$$\left|D_y^\alpha R_{2m+2}\right| < K|x|^{-n-2}, \tag{33}$$

whence we obtain

$$\|R_{2m+2}\|_{L_2^{(m)}(y<2Lh)} \leq Kh^{-n/2+m}|x|^{-n-2}. \tag{34}$$

Since

$$G(x) * l_3\left(\frac{x}{h}\right) = \int G(x-y) l_3\left(\frac{y}{h}\right)\, dy = \left(G(x-y),\, l_3\left(\frac{y}{h}\right)\right);$$

(31) follows from (26)–(30) and (34). The lemma is proved.

Theorem 2 is obtained from Lemma 2 by double summation, for example, if the sum (24) is estimated by means of the integral test.

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CITED LITERATURE

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3. S. L. Sobolev, DAN, 137, No. 3, 527 (1961).

Note: Figure translations are in progress. See original paper for figures.