ON THE CONTINUATION OF SOLUTIONS OF A ONE-DIMENSIONAL DIRAC SYSTEM

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Abstract

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MATHEMATICS

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ON THE CONTINUATION OF SOLUTIONS OF A ONE-DIMENSIONAL DIRAC SYSTEM

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1. Consider the solution of the system:

$$\begin{aligned} d\varphi_2/dx + p(x)\varphi_1 &= \lambda\varphi_1, \\ -d\varphi_1/dx + q(x)\varphi_2 &= \lambda\varphi_2, \end{aligned} \qquad x \geq 0; \tag{1}$$

$$\varphi_1(0) = 1, \qquad \varphi_2(0) = h, \tag{2}$$

where h is an arbitrary complex number.

Suppose that we are given a continuation of the coefficients $p(x)$ and $q(x)$ to the negative half-axis. Then the solution $\varphi = \{\varphi_1(x, \lambda), \varphi_2(x, \lambda)\}$ of system (1)–(2) can be continued to the negative half-axis by solving the corresponding Cauchy problem.

In the present note it is shown how to express the solution φ at the point $-x$ in the form of a linear operator applied to $\varphi = \{\varphi_1(t, \lambda), \varphi_2(t, \lambda)\}$ $(0 \leq t \leq x)$.

If the functions $p(x)$ and $q(x)$ are continued evenly, then, as is easy to see, the solutions of system (1) with initial conditions

$$\varphi_1(0, \lambda) = 1, \qquad \varphi_2(0, \lambda) = 0,$$

$$(\varphi_1(0, \lambda) = 0, \qquad \varphi_2(0, \lambda) = 1)$$

are continued as follows: $\varphi_1(x, \lambda)$ is even, $\varphi_2(x, \lambda)$ is odd ($\varphi_1(x, \lambda)$ is odd, $\varphi_2(x, \lambda)$ is even), and since these solutions form a fundamental system of solutions, it is thereby possible to continue, preserving continuity, any solution of system (1)–(2) for a given h.

In the case when $p(x)$ and $q(x)$ are continued arbitrarily, one may use transformation operators for the solution of the problem posed ([1]).

2. Denote by $\mathcal{E}_h[0,b)$, $0 < b \leq \infty$, the space of vector-functions $f(x) = \{f_1(x), f_2(x)\}$, continuously differentiable on the interval $[0,b)$ and satisfying the boundary condition

$$f_2(0) - hf_1(0) = 0. \tag{3}$$

Let

$$A \equiv \begin{pmatrix} p(x) & d/dx \\ -d/dx & q(x) \end{pmatrix}, \qquad B \equiv \begin{pmatrix} p_1(x) & d/dx \\ -d/dx & q_1(x) \end{pmatrix}; \tag{4}$$

where the functions $p(x), q(x), p_1(x)$, and $q_1(x)$ are continuous on the interval $[0,b)$.

The operator $X = X_{h_1,h_2;A,B}$ is called a **transformation operator** if it maps the space $\mathcal{E}_{h_1}$ into the space $\mathcal{E}_{h_2}$ and satisfies the following conditions:

$$\begin{array}{ll} 1) & AX = XB; \\ 2) & \text{there exists a continuous inverse operator } X^{-1}. \end{array} \tag{5}$$

It is known that the operator X can be given in the form (2)

$$X\{f(x)\} \equiv \begin{cases} \alpha(x)f_1(x) + \beta(x)f_2(x) + \displaystyle\int_0^x P(x,s)f_1(s)\,ds + \int_0^x R(x,s)f_2(s)\,ds, & (6) \\ \alpha(x)f_2(x) - \beta(x)f_1(x) + \displaystyle\int_0^x Q(x,s)f_1(s)\,ds + \int_0^x H(x,s)f_2(s)\,ds, & (7) \end{cases}$$

where the functions $\alpha(x)$ and $\beta(x)$ are determined by the formulas

$$\alpha(x) = \varkappa \sin\left\{-\frac{1}{2}\int_0^x [p(\tau) - p_1(\tau) + q(\tau) - q_1(\tau)]\,d\tau + \arcsin\frac{1}{\varkappa}\right\}, \tag{8}$$

$$\beta(x) = \varkappa \cos\left\{-\frac{1}{2}\int_0^x [p(\tau) - p_1(\tau) + q(\tau) - q_1(\tau)]\,d\tau + \arcsin\frac{1}{\varkappa}\right\}, \tag{9}$$

where

$$\varkappa = \{(1+h_1^2)(1+h_2^2)/(1+h_1h_2)^2\}^{1/2}. \tag{10}$$

From condition (5), by virtue of (6) and (7), for the functions $P(x,s)$, $R(x,s)$, $Q(x,s)$ and $H(x,s)$ it is not difficult to obtain the system of differential equations

$$\frac{\partial P(x,s)}{\partial s} + \frac{\partial H(x,s)}{\partial x} = \{q_1(s) - p(x)\}R(x,s),$$

$$\frac{\partial H(x,s)}{\partial s} + \frac{\partial P(x,s)}{\partial x} = \{q(x) - p_1(s)\}Q(x,s),$$

$$\frac{\partial Q(x,s)}{\partial s} - \frac{\partial R(x,s)}{\partial x} = \{q_1(s) - q(x)\}H(x,s),$$

$$\frac{\partial R(x,s)}{\partial s} - \frac{\partial Q(x,s)}{\partial x} = \{p(x) - p_1(s)\}P(x,s),$$

with the characteristic conditions

$$P(x,x) - H(x,x) = \alpha'(x) + \beta(x)\{p(x) - q_1(x)\},$$

$$R(x,x) + Q(x,x) = \beta'(x) - \alpha(x)\{p(x) - p_1(x)\},$$

and the initial conditions

$$R(x,0) - h_1 P(x,0) = 0,$$

$$H(x,0) - h_1 Q(x,0) = 0.$$

3. Let us now consider the application of transformation operators to the continuation of solutions of problem (1)–(2). Suppose that the coefficients of system (1)–(2), i.e., the functions $p(x)$ and $q(x)$, have been continued arbitrarily onto the negative half-axis. Put

$$A \equiv \begin{pmatrix} p(-x) & d/dx \\ -d/dx & q(-x) \end{pmatrix}, \qquad B \equiv \begin{pmatrix} p(x) & d/dx \\ -d/dx & q(x) \end{pmatrix} \quad (x \geqslant 0). \tag{4'}$$

Denote by $\varphi^+ = \{\varphi_1(x,\lambda), \varphi_2(x,\lambda)\}$ the solution of system (1)–(2) for $x \geqslant 0$, and put

$$\varphi^- = \{\varphi_1(-x,\lambda), \varphi_2(-x,\lambda)\} = X\{\varphi^+\}. \tag{11}$$

Equality (11), by virtue of the definition of the operator X, i.e., (6) and (7), is equivalent to the two equalities

$$\varphi_1(-x,\lambda) = \alpha(x)\varphi_1(x,\lambda) + \beta(x)\varphi_2(x,\lambda) + \int_0^x P(x,s)\varphi_1(s,\lambda)\,ds +$$

$$+ \int_0^x R(x,s)\varphi_2(s,\lambda)\,ds, \tag{12}$$

$$\varphi_2(-x,\lambda) = \alpha(x)\varphi_2(x,\lambda) - \beta(x)\varphi_1(x,\lambda) + \int_0^x Q(x,s)\varphi_1(s,\lambda)\,ds +$$

$$+ \int_0^x H(x,s)\varphi_2(s,\lambda)\,ds. \tag{13}$$

Let us show that the functions $\varphi_1(-x,\lambda)$ and $\varphi_2(-x,\lambda)$ satisfy system (1) on the negative half-axis. Indeed, using equality (5) and system (1), we find the equality

$$A\{\varphi^-\} = AX\{\varphi^+\} = XB\{\varphi^+\} = X\{\lambda\varphi^+\} = \lambda X\{\varphi^+\} = \lambda\varphi^-,$$

which, by the definition of the operator A (see (4′)), is equivalent to the system of two equations

$$d\varphi_2(-x,\lambda)/dx + p(-x)\varphi_1(-x,\lambda) = \lambda\varphi_1(-x,\lambda),$$

$$-d\varphi_1(-x,\lambda)/dx + q(-x)\varphi_2(-x,\lambda) = \lambda\varphi_2(-x,\lambda),$$

which coincides with system (1) for the functions $\varphi_1(-x,\lambda)$ and $\varphi_2(-x,\lambda)$.

Let us now show that this continuation is continuous. Indeed, putting $x = +0$ in equalities (12) and (13), we obtain

$$\varphi_1(-0,\lambda) = \alpha(0)\varphi_1(+0,\lambda) + \beta(0)\varphi_2(+0,\lambda), \tag{14}$$

$$\varphi_2(-0,\lambda) = \alpha(0)\varphi_2(+0,\lambda) - \beta(0)\varphi_1(+0,\lambda). \tag{15}$$

Further, in our case $h_1 = h_2 = h$, whence from equality (10) we find that $\chi = 1$. Then from equalities (8) and (9) it follows that $\alpha(0) = 1$, $\beta(0) = 0$. Therefore the continuity of the continuation follows from equalities (14) and (15).

Let us show that if the coefficients of system (1) are continued to the negative half-axis continuously, i.e. if

$$p(-0) = p(+0), \qquad q(-0) = q(+0), \tag{16}$$

then not only the continuation of the solution is continuous, but also its first derivative. Indeed, putting in the equality

$$A\{\varphi^-\} = XB\{\varphi^+\}$$

$x = +0$, we obtain

$$\begin{gathered} d\varphi_2(-0,\lambda)/dx + p(-0)\varphi_1(-0,\lambda) = \\ = \alpha(0)\{d\varphi_2(+0,\lambda)/dx + p(+0)\varphi_1(+0,\lambda)\} + \\ +\beta(0)\{-d\varphi_1(+0,\lambda)/dx + q(+0)\varphi_2(+0,\lambda)\}, \end{gathered} \tag{17}$$

$$\begin{gathered} -d\varphi_1(-0,\lambda)/dx + q(-0)\varphi_2(-0,\lambda) = \\ = \alpha(0)\{-d\varphi_1(+0,\lambda)/dx + q(+0)\varphi_2(+0,\lambda)\} - \\ -\beta(0)\{d\varphi_2(+0,\lambda)/dx + p(+0)\varphi_1(+0,\lambda)\}. \end{gathered} \tag{18}$$

Then, by virtue of equalities (16), $\alpha(0) = 1$, $\beta(0) = 0$, and the continuity of the continuation, it follows from equalities (17) and (18) that

$$d\varphi_1(-0,\lambda)/dx = d\varphi_1(+0,\lambda)/dx, \qquad d\varphi_2(-0,\lambda)/dx = d\varphi_2(+0,\lambda)/dx,$$

i.e. the continuity of the first derivative of the continuation.

4. The method indicated above can also be applied to the problem of continuing solutions of the nonstationary Dirac system.

Consider the mixed problem

$$\left.\begin{aligned} i\,\partial u_1/\partial t + \partial u_2/\partial x + p(x)u_1 = 0, \\ i\,\partial u_2/\partial t - \partial u_1/\partial x + q(x)u_2 = 0, \end{aligned}\right\} \quad x \geqslant 0; \tag{19}$$

$$u_1(x,0) = f_1(x), \qquad u_2(x,0) = f_2(x), \tag{20}$$

$$u_2(0,t) - hu_1(0,t) = 0. \tag{21}$$

Suppose that an extension of the coefficients $p(x)$ and $q(x)$ to the negative half-axis is given to us. Denote by $U(x,t) = \{u_1(x,t), u_2(x,t)\}$ the solution of problem (19)–(20)–(21), and put

$$u(x,t) = \begin{cases} U(x,t), & x \geqslant 0, \\ X\{U(x,t)\}, & x < 0, \end{cases}$$

where the operator X, as before, is defined by formulas (6) and (7). It is not difficult to verify that the function $u(x,t)$ is a solution of the pure Cauchy problem for system (19) with the initial conditions

$$U(x,0) = \begin{cases} f(x) = \{f_1(x), f_2(x)\}, & x \geqslant 0, \\ X\{f(x)\}, & x < 0. \end{cases}$$

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CITED LITERATURE

1. B. M. Levitan, DAN, **146**, No. 1, 30 (1962).
2. F. Prats, J. S. Toll, *Phys. Rev.*, **113**, No. 1, 363 (1959).

Note: Figure translations are in progress. See original paper for figures.