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Abstract

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ON SPACES WITH A MEASURE OF CONTINUAL WEIGHT

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By a **space with a measure** in this paper is meant a collection $\{E, Q, m\}$, where E is a set, Q is a σ-algebra of subsets of E, and m is a nonnegative countably additive function defined on Q, such that the following conditions are satisfied: 1) $mE = 1$; 2) for any two points $x, y \in E$ there is a set $A \in Q$ such that $x \in A,\ y \notin A$; 3) if $B \subset A$, where $A \in Q$ and $mA = 0$, then $B \in Q$.

The **weight of a space with a measure** is called the least of the numbers τ for which there exists a system Σ of sets from Q, of cardinality τ, such that: 1) for any two points $x, y \in E$ there is a set $A \in \Sigma$ for which $x \in A,\ y \notin A$; 2) every set $A \in Q$ has the form $A = A_1 \cup A_0$, where A_1 belongs to the smallest σ-algebra containing Σ, and A_0 is contained in a null set belonging to this σ-algebra.

Let $\{E, Q, m\}$ be a space with a measure and let E' be a subset of E of outer measure one. The space with a measure $\{E', Q', m'\}$, where $Q' = Q \cap E'$, and $m'(A \cap E') = mA$, will be called a **subspace** of the space $\{E, Q, m\}$.

A space with a measure $\{E, Q, m\}$ will be called **maximal** if: 1) every space with a measure having the same metric structure and the same weight as $\{E, Q, m\}$ is isomorphic to some subspace of the space $\{E, Q, m\}$; 2) if the space $\{E, Q, m\}$ is isomorphic to a subspace of some space with a measure of the same weight as $\{E, Q, m\}$, then this space is isomorphic to $\{E, Q, m\}$. From the definition of a maximal space it follows that, in the class of spaces with a measure of one weight and one metric structure, the maximal space, if it exists, is unique.

From the results of V. A. Rokhlin' s work (1) it follows that, in the class of spaces of countable weight with one metric structure, a maximal space exists and is the Lebesgue space. In the general case it remains unknown whether a maximal space exists for every weight and every metric structure. However, for continual weight the answer is positive; namely, the following holds.

Theorem 1. *In the class of spaces of continual weight with one metric structure there exists a maximal space.*

For a homogeneous metric structure this maximal space is the direct product of a continual number of unit intervals with the ordinary Lebesgue measure. In the general case, for an arbitrary metric structure, the maximal space is a product, generally speaking not direct, of a continual number of Lebesgue spaces.

We note that two products of the same number of Lebesgue spaces, having the continuum of points, are isomorphic if their metric structures are isomorphic.

Let $\{E, Q, m\}$ be a space with a measure. An **approximating compact class** of this space is a class Φ of nonempty subsets of E satisfying the requirements: 1) every countable centered system of sets from Φ has a nonempty intersection, and this

the intersection belongs to Φ; 2) for every set $A \in Q$ and every $\varepsilon > 0$ there exist a set $F \in \Phi$ and a set $B \in Q$ such that $B \subset F \subset A$ and $m(A \setminus B) < \varepsilon$. An approximating compact class is called a **bicompact class** if it satisfies the additional condition; 3) every centered system of sets from Φ has a nonempty intersection.

A measure space having an approximating compact class is called a **space with a compact measure** ([2]).

Maximal spaces of continual weight are spaces with a compact measure, but, generally speaking, not conversely (on this point see ([3])). We shall consider here some properties of spaces with a compact measure.

In the paper of V. D. Erokhin ([4]) the following definition of spaces with a weakly compact measure is given. A measure space $\{E, Q, m\}$ is called a **space with a weakly compact measure** if on E one can define a regular σ-topology such that for every $\varepsilon > 0$ there exists a compact set K_ε for which every set from Q contained in $E \setminus K_\varepsilon$ has measure less than ε.

Theorem 2. *A space with a weakly compact measure of continual weight has a compact approximating class, i.e. is a space with a compact measure.*

We shall call a measurable set A of positive measure an **empty atom** if the measure of each of its measurable subsets is either zero or mA, and the intersection of all measurable subsets of A of positive measure is empty. A measure space having empty atoms does not have an approximating bicompact class. For spaces of continual weight the following holds.

Theorem 3. *In a space with a compact measure of continual weight, having no empty atoms, there exists an approximating bicompact class.*

The proof of these theorems is carried out with the aid of representations of Boolean algebras with measure. A **representation of an algebra with measure** $\{\mathfrak{B}, m\}$ is a collection $\{E, R, V\}$, where E is a set, R is an algebra of subsets of E, and V is a homomorphism of the algebra $\mathfrak{B}$ onto R such that: 1) for any two points $x, y \in E$ there is a set $A \in R$ for which $x \in A,\ y \notin A$; 2) if $b \in \mathfrak{B}$ and $mb > 0$, then the set Vb is nonempty.

The measure m induces a measure on the algebra R by the equality $mB = mb$, where $Vb = B$.

Every algebra with measure has a maximal (Stone) representation, which is denoted by $\{E_c, R_c, V_c\}$. Every other representation $\{E, R, V\}$ of this algebra can be regarded as part of the representation $\{E_c, R_c, V_c\}$, namely, $E \subset E_c$ and for $b \in \mathfrak{B}$, $Vb = V_{cb} \cap E$.

Suppose that the measure m extends to a countably additive measure defined on the smallest σ-algebra S containing R. Let $[R]$ be the smallest σ-algebra containing S and all subsets of null sets from S. We say that the measure space $\{E, [R], m\}$ is **generated by the representation** $\{E, R, V\}$ of the algebra with measure $\{\mathfrak{B}, m\}$. Every measure space is generated by some representation of an algebra with measure, and the algebra can be chosen so that its cardinality is equal to the weight of the space.

Let τ denote the cardinality of the continuum. Let $\{\mathfrak{B}, m\}$ be a continual algebra with measure. We shall call a **τ-system** of $\{\mathfrak{B}, m\}$ a system $\{\mathfrak{B}_\lambda, m_\lambda\}$ of two nondecreasing transfinite sequences: the sequence of countable subalgebras

$$\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \dots \subset \mathfrak{B}_\lambda \subset \dots$$

of $\mathfrak{B}$, and the sequence of null sets of the space $\{E_c, [R_c], m\}$

$$N_1 \subset N_2 \subset \dots \subset N_\lambda \subset \dots,$$

in which the index λ runs through the set of all transfinite numbers less than ω_τ, the first transfinite number of cardinality τ, and which satisfy the conditions: 1) $\bigcup \mathfrak{B}_\lambda = \mathfrak{B}$; 2) if λ is a limit ···

transfinite number, then $\bigcup \mathfrak{B}_\nu = \mathfrak{B}_\lambda$ and $\bigcap_{\nu<\lambda} N_\nu = N_\lambda$; 3) N_λ belongs to the smallest σ-algebra containing the algebra $V_c\mathfrak{B}_\lambda$.

For a system of sets Σ, denote by $\zeta(\Sigma)$ a partition such that each of its elements either is contained in, or has empty intersection with, any set in Σ, and for any two elements of $\zeta(\Sigma)$ there is a set in Σ containing one element and not containing the other.

A representation $\{E, R, V\}$ of a continual algebra with measure $\{\mathfrak{B}, m\}$ is called a k-representation if there exists a τ-system $\{\mathfrak{B}_\lambda, N_\lambda\}$ of the algebra with measure $\{\mathfrak{B}, m\}$ for which the following condition is satisfied: for every λ, E has nonempty intersection with every element of the partition $\zeta(V_c\mathfrak{B}_\lambda)$ lying in $E_c \setminus N_\lambda$.

A representation $\{E, R, V\}$ of a continual algebra with measure $\{\mathfrak{B}, m\}$ is called a τ-representation if there exists a τ-system $\{\mathfrak{B}_\lambda, N_\lambda\}$ of the algebra with measure $\{\mathfrak{B}, m\}$ for which the following conditions are satisfied: 1) for every λ, the set $N_{\lambda+1} \cap E$ intersects, in a set of cardinality 2^τ, every element of the partition $\zeta(V_c\mathfrak{B}_\lambda)$ lying in $E_c \setminus N_\lambda$; 2) all nonmeasurable points of the measure space $\{E_c, [R_c], m\}$ are contained in E.

Theorem 4. *A representation of a continual algebra with measure generates a space with compact measure if and only if it is a k-representation.*

Theorem 5. *A representation of a continual algebra with measure generates a measure space isomorphic to a product of a continual number of Lebesgue spaces if and only if it is a τ-representation.*

The proof of Theorem 1 is based on Theorem 5, and those of Theorems 2 and 3 on Theorem 4. At the same time, the following fact, which follows from Theorems 4 and 5, plays the principal role. If a space $\{E, Q, m\}$ of continual weight has a subspace which is a space with compact measure or a product of a continual number of Lebesgue spaces, then the space $\{E, Q, m\}$ itself is a space with compact measure or, respectively, a product of a continual number of Lebesgue spaces.

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Note: Figure translations are in progress. See original paper for figures.