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Abstract

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DISTRIBUTIVE PAIRS OF ELEMENTS IN A LATTICE

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§ 1. A.d.-pairs and m.d.-pairs of elements in a lattice

Let L be a lattice, $a, b \ldots$ its elements, and let $a \cup b$, $a \cap b$ denote, respectively, the lattice sum and intersection. A pair (a, b) of elements of L will be called an a.d.-pair (additively distributive pair) if

$$(a \cup b) \cap x = (a \cap x) \cup (b \cap x)$$

for every $x \in L$. The pair (a, b) will be called an m.d.-pair (multiplicatively distributive pair) if

$$(a \cap b) \cup x = (a \cup x) \cap (b \cup x)$$

for every $x \in L$. The pair (a, b) will be called a strict a.d.-pair (respectively, a strict m.d.-pair) if it is not an m.d.-pair (respectively, an a.d.-pair). In the familiar five-element nonmodular lattice $\{0, 1, a, b, c\}$, $a > b$, $a \cup c = b \cup c = 1$, $a \cap c = b \cap c = 0$, the pair (a, c) is a strict a.d.-pair, and (b, c) is a strict m.d.-pair. A pair (a, b) that is simultaneously an a.d.-pair and an m.d.-pair will be called an a.m.d.-pair. An ordered pair of elements (a, b) will always be an a.m.d.-pair; such a pair will be called trivial.

In the literature an a.d.-pair is usually called a distributive pair. M.d.-pairs and the relations between the a.d.- and m.d.-properties for pairs have almost not been considered. Note, however, the following result, due to J. von Neumann ([1], p. 95):

In a modular lattice, the sublattice generated by the elements x, y, z is distributive if and only if

$$(x \cup y) \cap z = (x \cap z) \cup (y \cap z).$$

It follows from this, in particular, that in a modular lattice there are no strict a.d.-pairs or m.d.-pairs.

In the present note we consider some relations between the a.d.- and m.d.-properties for pairs in a lattice and applications of the theory of such pairs to group lattices. In particular, we indicate an important class of groups characterized by the absence of a.d.-pairs, and we prove that in a periodic group with the normalizer condition there are no strict a.d.- and m.d.-pairs.

Lemma 1. (a, b) is an a.d.-pair in L if and only if it is an a.d.-pair in the principal ideal $(a \cup b] = \{x \mid x \leq a \cup b\}$.

Dually: (a, b) is an m.d.-pair in L if and only if it is an m.d.-pair in the dual principal ideal $[a \cap b) = \{x \mid x \geq a \cap b\}$.

Theorem 1. An a.d.-pair (a, b) in $(a \cup b]$ will be an m.d.-pair in the same ideal if and only if one of the following conditions is satisfied:

1°. The pairs (b, x) and (a, y) are a.d.-pairs in $(a \cup b]$, where

$$x = (a \cap b) \cup (a \cap u), \qquad y = (a \cap b) \cup (b \cap v);$$

u, v independently range over the ideal $(a \cup b]$.

2°. In $(a \cup b]$ there do not exist elements x, y such that either

$$a \cap b < x < y \leq a$$

and the elements $b, x, y, b \cup x = b \cup y, a \cap b$ form a five-element nonmodular sublattice, or

$$a \cap b < x < y \leq b$$

and the elements $a, x, y, a \cup x = a \cup y, a \cap b$ form a five-element nonmodular sublattice.

3°. For every $x \in (a \cup b]$ one has:

$$a \cap (b \cup x) = (a \cap b) \cup (a \cap x), \tag{1}$$

$$b \cap (a \cup x) = (b \cap a) \cup (b \cap x). \tag{2}$$

In other words, the relation

$$(a \cup b) \cap x = (a \cap x) \cup (b \cap x)$$

admits all possible permutations of the elements a, b, x.

Remark. In 3^0 one may consider x varying in the intervals $[a \cap b, a]$, $[a \cap b, b]$, respectively, for (1) and (2).

Theorem $1'$. An m.d.-pair (a, b) in $[a \cap b)$ will then and only then be an a.d.-pair in the same ideal when one of the conditions dual to the conditions of Theorem 1 is satisfied.

Theorem 2. If (a, b) is an a.d.-pair in L, then $(a \cap x, b \cap x)$ is also an a.d.-pair in L for every $x \in L$.

The dual assertion for an m.d.-pair is also valid.

Lemma 2. If (a, b) is an a.d.-pair in L, then every pair (a_1, b_1) such that $a_1 \geq a$, $b_1 \geq b$, $a \cup b = a_1 \cup b_1$, is an a.d.-pair in L.

The dual assertion for m.d.-pairs is also valid.

§ 2. Distributive pairs in the lattice of subgroups of a group

G is a group; $a, b, \ldots$ are its elements; $L(G)$ is the lattice of its subgroups; $A, B, \ldots$ are subgroups of the group G (elements of $L(G)$); $\cup, \cap$ are the lattice operations in $L(G)$ (in what follows, for brevity, we shall speak of lattice operations in the group G itself).

Let us recall the following classical result of Ore ((2), p. 17):

Subgroups A and B form an a.d.-pair in the group G if and only if the orders with respect to A and B of every element x of $A \cup B$ (but not of A and not of B) are finite and relatively prime.

Theorem 3. A pair (A, B) of relatively prime subgroups will be an a.d.-pair or an m.d.-pair (and consequently also an a.m.d.-pair) in $G = A \cup B$ if and only if $G = A \times B$ is a direct product and the orders of the elements of A and B are finite and relatively prime.

Remark. For an a.d.-pair the theorem was proved in (2), p. 19.

Theorem 4. A periodic group G is free of nontrivial a.d.-pairs if and only if it is covered by its Sylow subgroups (i.e., every element of it has primary order).

Proof. If G has no nontrivial a.d.-pairs, then it contains no cyclic subgroups of order pq, where p, q are distinct primes, as follows from the cited theorem of Ore. Consequently, G is covered by its Sylow subgroups. Conversely, if G is covered by its Sylow subgroups, then it is easy to see that there are no nontrivial a.d.-pairs, since Ore' s criterion is not satisfied.

Remark. Finite groups with a Sylow covering basis (s.b.-groups in the terminology of (3)) have been studied by many authors (Higman and others; see the bibliography in (3)). Theorem 4 gives a purely lattice-theoretic definition of this class of groups.

Lemma 3. Let $G = A \cup B$ and let $A \cap B = D$ be an invariant subgroup in G. Then the pair (A, B) is an m.d.-pair if and only if it is an a.d.-pair (and consequently also an a.m.d.-pair).

Proof. If (A, B) is an a.d.-pair in G, then $(A/D, B/D)$ is an a.d.-pair in the direct product

$$G/D = A/D \times B/D.$$

But then $(A/D, B/D)$ is an a.m.d.-pair in G/D by Theorem 3. Hence it follows from Lemma 1 that (A, B) is an m.d.-pair in G.

If, however, (A, B) is an m.d.-pair in G, then $(A/D, B/D)$ is an m.d.-pair in G/D, and consequently an a.m.d.-pair by Theorem 3. But then (A, B) is an a.d.-pair in G by Ore' s theorem.

Theorem 5. A p-group with the normalizer condition, in particular a finite p-group, is free of nontrivial a.d.-pairs and m.d.-pairs.

Proof. That every p-group is free of nontrivial a.d.-pairs follows directly from Theorem 4. Suppose now that (A, B) is an m.d.-pair. Then

$$A \cap B = D \neq e.$$

By virtue of the normalizer condition, there exist A_1 and B_1 such that $D \subset A_1 \subseteq A$, $D \subset B_1 \subseteq B$, and D is invariant in A_1 and B_1. By the preceding lemma, (A_1, B_1) will be a nontrivial a.d.-pair, which is impossible.

Lemma 4. In a periodic S-group (i.e., in a group that is the direct product of its Sylow subgroups), the intersection $(A \cap P, B \cap P)$ of an m.d.-pair (A, B) with a p-Sylow subgroup P will be an m.d.-pair in P.

Proof. Introduce the notation: $P_A = P \cap A,\ P \cap B = P_B,\ D \cap P = P_D = P_A \cap P_B,\ D = A \cap B$. Let $G = P \times H,\ A = P_A \times H_A,\ B = P_B \times H_B$. Consider a subgroup X of P for which $P_D \subseteq X$ (see Lemma 1). Then

$$X \subseteq (X \cup P_A) \cap (X \cup P_B) \subseteq P \cap (X \cup A) \cap (X \cup B) =$$

$$= P \cap [X \cup (A \cap B)] = P \cap [X \cup (P_D \times H_D)] = P \cap (X \times H_D).$$

Since (X, H_D) is a commuting pair, the modular identity is satisfied for it. Therefore

$$P \cap (X \times H_D) = X \cup (P \cap H_D) = X,$$

since $P \cap H_D = e$. Now the equality

$$X \cup (P_A \cap P_B) = (X \cup P_A) \cap (X \cup P_B)$$

is easily obtained.

Theorem 6. *A periodic group with the normalizer condition contains no proper a.d.-pairs and no proper m.d.-pairs.*

Proof. Let G be a periodic group with the normalizer condition. As is known, $G = \prod P_i$, where the P_i are Sylow p_i-subgroups, and $\prod$ denotes the sign of the direct product.

1) Let (A, B) be an m.d.-pair in G. Put $P_{iA} = P_i \cap A,\ P_{iB} = P_i \cap B$. By Theorem 5 and Lemma 4, either $P_{iA} \subseteq P_{iB}$ or $P_{iB} \subseteq P_{iA}$. Let

$$H = \prod P_{iA}, \quad P_{iA} \supseteq P_{iB}; \qquad F = \prod P_{jB}, \quad P_{jB} \supset P_{jA}.$$

It is clear that $A \cup B = H \cup F = H \times F$. By Lemma 1 and Theorem 3, (H, F) is an a.d.-pair in G. Therefore the orders m and n of an element from $[(H \times F) \setminus H] \setminus F$, respectively relative to H and F, are finite and relatively prime. Now let $x \in [(A \cup B) \setminus A] \setminus B$, and let the orders of the element x relative to A and B be respectively m_1 and n_1. From $H \subseteq A$, $F \subseteq B$ it follows that $x \in [(H \times F) \setminus H] \setminus F$, and m_1 divides m, n_1 divides n, i.e., m_1 and n_1 are also relatively prime. Consequently, (A, B) is an a.d.-pair in G.

2) Let (A, B) be an a.d.-pair in G. According to Theorem 2, (P_{iA}, P_{iB}) is an a.d.-pair in G, and hence all the more so in P_i. By Theorem 5 it follows that either $P_{iA} \subseteq P_{iB}$ or $P_{iB} \subset P_{iA}$.

We now prove that (A, B) is an m.d.-pair in G. Form the following subgroups:

$$H = \prod P_{iA},\ P_{iA} \supseteq P_{iB}; \qquad F = \prod P_{jB},\ P_{jB} \supset P_{jA};$$

$$H_1 = \prod P_{iB},\ P_{iA} \supseteq P_{iB}; \qquad F_1 = \prod P_{jA},\ P_{jB} \supset P_{jA}.$$

It is clear that $A \cup B = H \times F$, $A = H \times F_1$, $B = H_1 \times F$. Let $\pi_2 = \pi(F)$ be the set of all prime numbers dividing the orders of elements of F, and let π_1 be the totality of all the remaining prime numbers dividing the orders of elements of the group G; in particular, $\pi_1 \supseteq \pi(H)$. Let R_1 and R_2 be respectively Sylow π_1- and π_2-subgroups of the group G, so that $G = R_1 \times R_2$. Let Y be a subgroup of G, and $Y \supseteq A \cap B$ (see Lemma 1). It is not hard to see that

$$Y \subseteq (Y \cup A) \cap (Y \cup B) =$$

$$= \{[(Y \cap R_1) \cup H] \times [(Y \cap R_2) \cup F_1]\} \cap \{[(Y \cap R_1) \cup H_1] \times$$
$$\times [(Y \cap R_2) \cup F]\}.$$

The pairs of direct factors standing in braces will be a.d.-pairs in G. Taking into account that

$$[(Y \cap R_1) \cup H] \cap [(Y \cap R_2) \cup F] = [(Y \cap R_1) \cup H_1] \cap [(Y \cap R_2) \cup F_1] = e$$

and that $H_1 \subseteq A \cap B$, $F_1 \subseteq A \cap B$, we obtain, after some calculation,

$$Y \cup (A \cap B) = (Y \cup A) \cap (Y \cup B),$$

as required.

Theorem 7. *If A, B are quasinormal subgroups in a group G (i.e., they permute with every subgroup of G) and (A, B) is an m.d.-pair in $A \cup B$, then (A, B) is an m.d.-pair in the whole group G.*

In the proof, the quasinormality of the subgroups and the modular identity for such subgroups are used.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.