



Soviet-era science, translated into English

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In the present paper we investigate the asymptotics of the solution of the following boundary-value problem.

1965

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Abstract

Full Text

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**ASYMPTOTICS OF THE SOLUTION OF
A BOUNDARY-VALUE PROBLEM FOR A
SECOND-ORDER ELLIPTIC EQUATION IN
DOMAINS IN WHICH ONE DIMENSION IS
SUFFICIENTLY SMALL IN COMPARISON
WITH THE OTHERS**

(Presented by Academician I. N. Vekua on 11 VII 1964)

In the present paper we investigate the asymptotics of the solution of the following boundary-value problem.

Let Q be a cylinder of height h , small in relation to the remaining dimensions. Denote the lateral surface of the cylinder by F and consider the following problem:

$$\mathcal{L}u \equiv \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial u}{\partial x_i} - bu = 0; \quad (1)$$

$$a_{nn} \frac{\partial u}{\partial x_n} + \sum_{i=1}^{n-1} a_{in} \frac{\partial u}{\partial x_i} + \frac{a_n}{2} u = \begin{cases} P & \text{when } x_n = h, \\ 0 & \text{when } x_n = 0; \end{cases} \quad (2)$$

$$u|_F = \Phi(s, x_n), \quad (3)$$

where $a_{ij} = a_{ji}$,

$$\sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \alpha \sum_{i=1}^n \xi_i^2, \quad \alpha = \text{const} > 0, \quad b \geq 0,$$

$P(x_1, x_2, \dots, x_{n-1})$ and $\Phi(s, x_n)$ are prescribed smooth functions.

Make the change of variables $x_n = th$. We have

$$\mathcal{L}u \equiv \frac{a_{nn}}{h^2} \frac{\partial^2 u}{\partial t^2} + \frac{1}{h} \left(\sum_{i=1}^{n-1} a_{in} \frac{\partial^2 u}{\partial x_i \partial t} + a_n \frac{\partial u}{\partial t} \right) +$$

$$+ \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} a_i \frac{\partial u}{\partial x_i} - bu = 0; \quad (1')$$

$$\frac{a_{nn}}{h} \frac{\partial u}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial u}{\partial x_i} + \frac{a_n}{2} u = \begin{cases} P & \text{when } t = 1, \\ 0 & \text{when } t = 0; \end{cases} \quad (2')$$

$$u|_F = \Phi(s, th). \quad (3')$$

We shall seek an asymptotic representation of the solution of the posed problem in the form $u = \tilde{w}_n + \tilde{v}_n + z_n$, where the function \tilde{w}_n will be determined by the first iterative process, \tilde{v}_n by the second, and z_n is the residual term.

First iterative process. Splitting the operator \mathcal{L} in (1') corresponds to a recurrent process, which is obtained if the approximate solution of equation (1') is sought in the form

$$\tilde{w}_n = w_{-1}/h + w_0 + hw_1 + \dots + h^n w_n + h^{n+1} w_{n+1}. \quad (4)$$

Substituting the expression for \tilde{w}_n from (4) into (1'), (2') and comparing terms with equal powers of h , we obtain

$$\partial^2 w_{-1} / \partial t^2 = 0; \quad \partial w_{-1} / \partial t = 0 \quad \text{when } t = 1; \quad \partial w_{-1} / \partial t = 0 \quad \text{when } t = 0; \quad (5)$$

$$a_{nn} \frac{\partial^2 w_0}{\partial t^2} = - \sum_{i=1}^{n-1} a_{in} \frac{\partial^2 w_{-1}}{\partial x_i \partial t} - a_n \frac{\partial w_{-1}}{\partial t}; \quad (6)$$

$$a_{nn} \frac{\partial w_0}{\partial t} + \sum_{i=1}^{n-1} a_{in} \frac{\partial w_{-1}}{\partial x_i} + \frac{a_n}{2} w_{-1} = \begin{cases} 0 & \text{when } t = 1, \\ 0 & \text{when } t = 0; \end{cases} \quad (7)$$

$$a_{nn} \frac{\partial^2 w_1}{\partial t^2} = - \sum_{i=1}^{n-1} a_{in} \frac{\partial^2 w_0}{\partial x_i \partial t} - a_n \frac{\partial w_0}{\partial t} - \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 w_{-1}}{\partial x_i \partial x_j} - \sum_{i=1}^{n-1} a_i \frac{\partial w_{-1}}{\partial x_i} + bw_{-1}; \quad (8)$$

$$a_{nn} \frac{\partial w_1}{\partial t} + \sum_{i=1}^{n-1} a_{in} \frac{\partial w_0}{\partial x_i} + \frac{a_n}{2} w_0 = \begin{cases} P & \text{for } t = 1, \\ 0 & \text{for } t = 0; \end{cases} \quad (9)$$

$$a_{nn} \frac{\partial^2 w_k}{\partial t^2} = - \sum_{i=1}^{n-1} a_{in} \frac{\partial^2 w_{k-1}}{\partial x_i \partial t} - a_n \frac{\partial w_{k-1}}{\partial t} - \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 w_{k-2}}{\partial x_i \partial x_j} - \sum_{i=1}^{n-1} a_{ij} \frac{\partial w_{k-2}}{\partial x_i} + b w_{k-2}; \quad (10)$$

$$a_{nn} \frac{\partial w_k}{\partial t} + \sum_{i=1}^{n-1} a_{in} \frac{\partial w_{k-1}}{\partial x_i} + \frac{a_n}{2} w_{k-1} = \begin{cases} 0 & \text{for } t = 1, \\ 0 & \text{for } t = 0, \end{cases} \quad k = 2, 3, \dots \quad (11)$$

From (5) it is clear that $w_{-1} = \bar{w}_{-1}(x_1, \dots, x_{n-1})$, i.e., the solution of problem (5) does not depend on t (on x_n).

Taking this into account, from (6) and (7) we find that

$$w_0 = \bar{w}_0 - t \left(\sum_{i=1}^{n-1} \frac{a_{in}}{a_{nn}} \frac{\partial \bar{w}_{-1}}{\partial x_i} + \frac{a_n}{2a_{nn}} \bar{w}_{-1} \right), \quad \text{where } \bar{w}_0 = \bar{w}_0(x_1, \dots, x_{n-1}).$$

We note that all the functions \bar{w}_k are to be determined subsequently. Substituting the expressions for w_{-1} and w_0 into (8) and (9), we obtain

$$\begin{aligned} a_{nn} \frac{\partial^2 w_1}{\partial t^2} &= \sum_{i,j=1}^{n-1} \frac{a_{in} a_{jn}}{a_{nn}} \frac{\partial^2 \bar{w}_{-1}}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} \frac{3 a_{in} a_n}{2 a_{nn}} \frac{\partial \bar{w}_{-1}}{\partial x_i} + \frac{a_n^2}{2a_{nn}} \bar{w}_{-1} \\ &\quad - \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 \bar{w}_{-1}}{\partial x_i \partial x_j} - \sum_{i=1}^{n-1} a_i \frac{\partial \bar{w}_{-1}}{\partial x_i} + b \bar{w}_{-1}, \\ a_{nn} \frac{\partial w_1}{\partial t} + \sum_{i=1}^{n-1} a_{in} \frac{\partial w_0}{\partial x_i} + \frac{a_n}{2} w_0 &= \begin{cases} P & \text{for } t = 1, \\ 0 & \text{for } t = 0. \end{cases} \end{aligned}$$

It is obvious that the adjoint homogeneous problem corresponding to problem (8) and (9) is a problem of the form (5), which has a nonzero solution independent of t . Denote it by z_0 . Then the solvability condition for the last problem is written as follows:

$$\int_0^1 a_{nn} \frac{\partial^2 w_1}{\partial t^2} z_0 dt = a_{nn} \frac{\partial w_1}{\partial t} z_0 \Big|_{t=0}^{t=1}.$$

Hence

$$l_1(\bar{w}_{-1}) \equiv - \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 \bar{w}_{-1}}{\partial x_i \partial x_j} - \sum_{i=1}^{n-1} \left(a_i - \frac{a_{in} a_n}{2a_{nn}} \right) \frac{\partial \bar{w}_{-1}}{\partial x_i} + \left(b - \frac{a_n^2}{4a_{nn}} \right) \bar{w}_{-1} = P.$$

Consequently,

$$w_1 = \bar{w}_1 - t \left(\sum_{i=1}^{n-1} a_{in} \frac{\partial \bar{w}_0}{\partial x_i} + \frac{a_n}{2} \bar{w}_0 \right) + \frac{t^2}{2!} \left(P + \sum_{i,j=1}^{n-1} \frac{a_{in} a_{jn}}{a_{nn}} \frac{\partial^2 \bar{w}_{-1}}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} \frac{a_{in} a_n}{a_{nn}} \frac{\partial \bar{w}_{-1}}{\partial x_i} + \frac{a_n^2}{4a_{nn}} \bar{w}_{-1} \right).$$

Substituting the expressions for w_0 and w_1 into (10) and (11) for $k = 2$ and writing the solvability condition for the resulting problem, we find

$$\begin{aligned} l_2(\bar{w}_0) &\equiv - \sum_{i,j=1}^{n-1} a_{ij} \frac{\partial^2 \bar{w}_0}{\partial x_i \partial x_j} - \sum_{i=1}^{n-1} \left(a_i - \frac{a_{in} a_n}{2} \right) \frac{\partial \bar{w}_0}{\partial x_i} + \left(b + \frac{a_n^2}{4} \right) \bar{w}_0 = \\ &= \frac{a_n}{4} P - \sum_{i,j,k=1}^{n-1} \frac{a_{ij} a_{kn}}{a_{nn}} \frac{\partial^3 \bar{w}_{-1}}{\partial x_i \partial x_j \partial x_k} + \sum_{i,j=1}^{n-1} \frac{a_{in} a_{jn} a_n - a_{ij} a_n - 2a_i a_{jn}}{4a_{nn}} \frac{\partial^2 \bar{w}_{-1}}{\partial x_i \partial x_j} + \\ &\quad + \sum_{i=1}^{n-1} \frac{a_{in} a_n^2 - a_{in} a_n + 2a_{in} b}{4a_{nn}} \frac{\partial \bar{w}_{-1}}{\partial x_i} + \frac{a_n b}{4a_{nn}} \bar{w}_{-1}. \end{aligned}$$

Consequently, $w_2 = \bar{w}_2 + f_2$, where f_2 is a known function, if the functions \bar{w}_{-1} , \bar{w}_0 , and \bar{w}_1 have been determined.

Continuing the process in this way, we obtain that $w_k = \bar{w}_k + f_k$, where f_k are known functions, and the functions \bar{w}_k are determined from the following problems:

$$l_1(\bar{w}_{-1}) = P, \quad \bar{w}_{-1}|_{F_1} = 0, \quad l_2(\bar{w}_k) = \varphi_k, \quad k = 0, 1, \dots \quad (12)$$

where φ_k are known functions expressed in terms of previously determined functions.

The boundary conditions for the latter equations will be formulated below. Obviously, generally speaking, the functions w_k do not satisfy the boundary conditions on F . Therefore, to w_k we add functions of boundary-layer type in

such a way that the resulting sum satisfies all the boundary conditions. These functions are constructed by a second iteration process.

Second iteration process. In order to describe the second iteration process, in a sufficiently small neighborhood of F we introduce local coordinates (ρ, y) , where ρ is the distance along the normal to F , $y = y(t, y_1, \dots, y_{n-1})$ is the coordinate of a point on F , and make the change of variables $\rho = ht$. In the new coordinates equation (1') is written as

$$\mathcal{L}v \equiv \frac{1}{h^2}M_0v + \frac{1}{h}M_1v + M_2v, \quad (1'')$$

where

$$M_0 \equiv a_{nn}\partial^2/\partial t^2 + A\partial^2/\partial t \partial \tau + B\partial^2/\partial \tau^2, \quad M_1 \equiv \sum_{i=1}^{n-2} a_{in} \frac{\partial^2}{\partial t \partial y_i} + \frac{\partial}{\partial t} + a \frac{\partial}{\partial \tau},$$

$$M_2 \equiv \sum_{i,j=1}^{n-2} a_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{n-2} a_i \frac{\partial}{\partial y_i} - b.$$

We seek the boundary-layer solution of (1'') in the form

$$\tilde{v}_n = v_{-1}/h + v_0 + hv_1 + \dots + h^n v_n + h^{n+1} v_{n+1}, \quad (13)$$

so that

$$a_{nn} \frac{\partial v_{-1}}{\partial t} + A \frac{\partial v_{-1}}{\partial \tau} = \begin{cases} 0 & \text{for } t = 1, \\ 0 & \text{for } t = 0; \end{cases}$$

$$a_{nn} \frac{\partial v_k}{\partial t} + A \frac{\partial v_k}{\partial \tau} + \sum_{i=1}^{n-2} a_{in} \frac{\partial v_{k-1}}{\partial y_i} + \frac{a_n}{2} v_{k-1} = \begin{cases} 0 & \text{for } t = 1, \\ 0 & \text{for } t = 0; \end{cases} \quad (2'')$$

$$v_k|_{t=0} = -w_k|_F + \Phi_k, \quad k = -1, 0, 1, \dots, \quad (3'')$$

where Φ_k are the coefficients in the expansion of the function $\Phi(s, th)$ in powers of h .

Substituting the expression for \tilde{v}_n from (13) into (1''), comparing coefficients of equal powers of h , and taking into account (2'') and (3''), we obtain

$$M_0 v_{-1} = 0; \quad a_{nn} \frac{\partial v_{-1}}{\partial t} + A \frac{\partial v_{-1}}{\partial \tau} = \begin{cases} 0 & \text{for } t = 1, \\ 0 & \text{for } t = 0; \end{cases} \quad v_{-1}|_{\tau=0} = -w_{-1}|_F = 0,$$

i.e. $v_{-1} = 0$;

$$a_{nn} \frac{\partial^2 v_0}{\partial t^2} + A \frac{\partial^2 v_0}{\partial t \partial \tau} + B \frac{\partial^2 v_0}{\partial \tau^2} = 0; \quad (14)$$

$$a_{nn} \frac{\partial v_0}{\partial t} + A \frac{\partial v_0}{\partial \tau} = \begin{cases} 0 & \text{for } t = 1, \\ 0 & \text{for } t = 0; \end{cases} \quad (15)$$

$$v_0|_{\tau=0} = -w_0|_F + \Phi_0 = \Psi_0. \quad (16)$$

Finally,

$$M_0 v_k = -M_1 v_{k-1} - M_2 v_{k-2}; \quad (17)$$

$$a_{nn} \frac{\partial v_k}{\partial t} + A \frac{\partial v_k}{\partial \tau} + \sum_{i=1}^{n-2} a_{in} \frac{\partial v_{k-1}}{\partial y_i} + \frac{a_n}{2} v_{k-1} = \begin{cases} 0 & \text{for } t = 1, \\ 0 & \text{for } t = 0; \end{cases} \quad (18)$$

$$v_k|_{t=0} = \Phi_k - w_k|_F = \Psi_k, \quad k = 1, 2, \dots \quad (19)$$

The boundary conditions for determining the function \bar{w}_k are specified so that $\Psi_k(0) = 0$ and $\Psi_k(1) = 0$, $k = 0, 1, \dots$

Obviously, if Ψ_0 is a smooth function, then the solution v_0 of problem (14), (15), and (16) will also be a smooth function, except at the points $(0, 0)$ and $(0, 1)$. By methods of functional analysis it is proved that v_0 has a first derivative from $L_2(0, 1)$.

Lemma 1. *Let the function Ψ_0 be such that $\Psi(0) = 0$ and $\Psi(1) = 0$. Then problem (14), (15), (16) has a unique solution, which is representable in the form*

$$v_0 = \sum_{k=1}^{\infty} c_k e^{-\gamma_k \tau} \omega_k(t),$$

where $\gamma > 0$; $\omega_{-k}(t)$ are the eigenfunctions of the problem

$$a_{nn} \omega_k''(t) + A \lambda_k \omega_k'(t) + B \lambda_k^2 \omega_k(t) = 0,$$

$$a_{nn} \omega_k'(0) + A \lambda_k \omega_k(0) = 0, \quad a_{nn} \omega_k'(1) + A \lambda_k \omega_k(1) = 0,$$

which correspond to nonpositive values λ_k .

Now let us find v_i ($i = 1, 2, \dots$). We define them as the solution of the equation $M_0 v_i = -M_1 v_{i-1} - M_2 v_{i-2}$ under the conditions that v_i is a function of boundary-layer type and satisfies conditions (18) and (19). Assuming that v_j , for $j < i$, is a function of boundary-layer type (this condition is fulfilled for $j = 0$), we find that all the functions v_i are also functions of boundary-layer type.

Let us multiply v_i by smoothing functions and again denote the resulting functions by v_i . Then for the solution of problem (1), (2), (3) we obtain the expansion

$$u = \sum_{i=-1}^{n+1} h^i w_i + \sum_{j=0}^{n+1} h^j v_j + z_n. \quad (20)$$

Obviously, $\mathcal{L}z_n = h^{n+1} \mathcal{L}g = g_1$. Moreover,

$$\frac{1}{h} a_{nn} \frac{\partial z_n}{\partial t} + \sum_{i=1}^{n-1} a_{in} \frac{\partial z_n}{\partial x_i} + \frac{a_n}{2} z_n = \begin{cases} 0 & \text{for } t = 1, \\ 0 & \text{for } t = 0, \end{cases} \quad z_n|_F = 0.$$

Lemma 2. *If $P(x_1, \dots, x_{n-1})$ and $\Phi(s, x_n)$ are sufficiently smooth functions, then the estimate*

$$\frac{1}{h^2} \left\| \frac{\partial z_n}{\partial t} \right\|^2 + \sum_{i=1}^{n-1} \left\| \frac{\partial z_n}{\partial x_i} \right\|^2 + \|z_n\|^2 \leq c \|g_1\|^2,$$

holds, where c is a constant independent of h , and the norm is understood in the sense of the metric of the space L_2 .

Thus, the following has been proved.

Theorem. *Let $P(x_1, \dots, x_{n-1})$ and $\Phi(s, x_n)$ be sufficiently smooth functions. Then problem (1), (2), (3) has a unique solution, the first and second iterative processes are feasible, for the solution of the problem the asymptotic expansion (20) holds, and the remainder tends to zero as $h \rightarrow 0$ like h^{n+1} in the metric $L_2(Q)$.*

The author expresses gratitude to L. A. Lyusternik and M. I. Vishik for formulating the problem and for advice in its solution.

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Received
26 VI 1964

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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