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## Abstract

## Full Text

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## THEORY OF ELASTICITY

V. K. FEDYANIN

## ON A PROBLEM IN THE THEORY OF ELASTICITY

*(Presented by Academician A. A. Dorodnitsyn, 23 III 1965)*

1. Let us pose the problem of finding a solution of the equations of the theory of elasticity for the half-space ( $z \geq 0$ ), which has, at a certain point located at a distance  $h$  from the boundary surface ( $z = 0$ ), a singularity of the type of a center (of compression or expansion). We shall assume that the boundary condition

$$p_z(r, 0) = 0; \quad (1)$$

is satisfied;  $p_z$  is the  $z$ -component of the pressure, and all quantities are functions of  $(r, z)$ : since there is cylindrical symmetry, it is convenient to consider the problem in cylindrical coordinates; the plane  $z = 0$  is assumed to coincide with the boundary, and the  $z$ -axis is directed downward and passes through the point  $(0, h)$ . Condition (1) leads to the following boundary conditions for the components of the stress tensor  $\sigma_{ik}$ :

$$\text{a) } \sigma_{zz}(r, 0) = 0, \quad \text{b) } \sigma_{zr}(r, 0) = 0. \quad (2)$$

We note that the solution of the same problem for the case

$$p_z(r, 0) = f(r) \quad (3)$$

can be carried out quite analogously (with the corresponding modification of (2)), as will be seen from what follows.

2. A solution with a singularity of center type satisfying condition (2b) can be obtained by taking the displacement vector in the form

$$u_z^0 = A_0 \left( \frac{z-h}{R_1^3} + \frac{z+h}{R_2^3} \right), \quad u_r^0 = A_0 r \left( \frac{1}{R_1^3} + \frac{1}{R_2^3} \right), \quad (4)$$

$$R_1^2 = r^2 + (z - h)^2; \quad R_2^2 = r^2 + (z + h)^2; \quad A_0 \text{ is a constant.}$$

In obtaining (4), we have used the known solution of the elasticity equations for an infinite medium <sup>(1)</sup>, taking two centers (at the points  $(0, h)$  and  $(0, -h)$ ). Convenient formulas for transforming the components of the stress tensor and displacement vector are given in <sup>(16)</sup>.

For the components of the stress tensor, (4) gives

$$\begin{aligned} \sigma_{zz}^0 &= -4\mu A_0 \left\{ \frac{1}{R_1^3} \left( 1 - \frac{3r^2}{2R_1^2} \right) + \frac{1}{R_2^3} \left( 1 - \frac{3r^2}{2R_2^2} \right) \right\}, & \sigma_{\theta\theta}^0 &= 2\mu A_0 \left( \frac{1}{R_1^3} + \frac{1}{R_2^3} \right), \\ \sigma_{rr}^0 &= 2\mu A_0 \left\{ \frac{1}{R_1^3} \left( 1 - 3\frac{r^2}{R_1^2} \right) + \frac{1}{R_2^3} \left( 1 - 3\frac{r^2}{R_2^2} \right) \right\}, & (5) \\ \sigma_{rz}^0 &= -6\mu A_0 \left\{ \frac{r(z-h)}{R_1^5} + \frac{r(z+h)}{R_2^5} \right\}, \end{aligned}$$

where  $\theta$  is the polar angle in the plane  $((x, y) : z = 0)$ , and  $\mu$  is the shear modulus (for notation, see, for example, <sup>(1)</sup>). The expressions (5) make it possible to satisfy one of the boundary conditions  $[\sigma_{rz}(r, 0) = 0]$ , but for  $\sigma_{zz}(r, 0)$  we have

$$\sigma_{zz}^0(r, 0) = -\frac{8\mu A_0}{\rho^3} \left( 1 - \frac{3r^2}{2\rho^2} \right), \quad \rho^2 = r^2 + h^2. \quad (6)$$

Thus, we shall be able to satisfy all the stated requirements if we construct a solution  $(u_r^1, u_z^1)$  having no singularities in the region  $z \geq 0$ , for which the following boundary conditions are satisfied:

$$\text{a) } \sigma_{zz}^1(r, 0) = \frac{8\mu A_0}{\rho^3} \left( 1 - \frac{3r^2}{2\rho^2} \right); \quad \text{b) } \sigma_{zr}^1(r, 0) = 0. \quad (7)$$

We shall solve this problem by means of the Laplace-transform method <sup>(2)</sup>.

**3.** A solution having no singularities for  $z \geq 0$  is obtained from Hankel transforms of the form <sup>(26)</sup>

$$G(\xi, z) = [A(\xi) + zB(\xi)]e^{-\xi z}, \quad (8)$$

where  $\xi$  is the transform parameter  $(0 \leq \xi < \infty)$ ;  $A(\xi)$ ,  $B(\xi)$  are unknown functions, for the determination of which we shall use (7). Using the expressions for  $\sigma_{ik}$  in terms of  $G(\xi, z)$ , given in <sup>(26)</sup>, and substituting (8), we obtain

$$\begin{aligned}\sigma_{zz} &= \int_0^\infty 2\xi^3 e^{-\xi z} [(\lambda + \mu)\xi A + (\mu + \xi z(\lambda + \mu))B] J_0(\xi r) d\xi, \\ \sigma_{rz} &= \int_0^\infty 2\xi^3 e^{-\xi z} [(\lambda + \mu)\xi A + (-\lambda + \xi z(\lambda + \mu))B] J_1(\xi r) d\xi;\end{aligned}\quad (9)$$

$\lambda, \mu$  are the Lamé elastic constants (see (1));  $J_k(x)$  is the Bessel function of order  $k$ . Taking (9) for  $z = 0$  and comparing with (7) by means of the Hankel-transform theorem, we obtain the following algebraic equations for  $A(\xi)$  and  $B(\xi)$ :

$$\begin{aligned}\mu B(\xi) + \xi(\lambda + \mu)A(\xi) &= \int_0^\infty \frac{2\mu A_0}{\xi^2} \frac{r J_0(\xi r)}{\rho^3} \left(-1 + 3\frac{h^2}{\rho^2}\right) dr = j(\xi, h), \\ -\lambda B(\xi) + \xi(\lambda + \mu)A(\xi) &= 0.\end{aligned}\quad (10)$$

It is obvious that all the arguments carry over unchanged to the case (3), leading only to a complication of  $j(\xi, h)$  in (10).

The integral in (10) is readily evaluated <sup>(26,3)</sup>, and we finally obtain

$$j(\xi, h) = \frac{2\mu A_0}{\xi} e^{-h\xi}, \quad B(\xi) = \frac{\lambda e^{-h\xi}}{(\lambda + \mu)^2} \frac{A_0}{2\xi^2}, \quad A(\xi) = \frac{A_0 e^{-h\xi}}{2\xi(\lambda + \mu)}, \quad (11)$$

$$G(\xi, z) = \frac{A_0 e^{-\xi(h+z)}}{2\xi^2(\lambda + \mu)} \left[ \frac{\lambda}{\lambda + \mu} + \xi z \right].$$

Formulas (11), (9), and the corresponding expressions for  $\sigma_{rr}, \sigma_{\theta\theta}, u_r, u_z$ , given in <sup>(26)</sup>, make it possible without difficulty to obtain the desired expressions:

$$\begin{aligned}u_r^1 &= \frac{2A_0 r}{R_2^3} \left( \frac{\mu}{\lambda + \mu} - \frac{3z(z+h)}{R_2^2} \right), \\ u_z^1 &= -\frac{2A_0}{R_2^3} \left[ \frac{\lambda + 2\mu}{\lambda + \mu} (z+h) + 2z \left( 1 - \frac{3r^2}{2R_2^2} \right) \right], \\ \sigma_{zz}^1 &= \frac{8\mu A_0}{R_2^3} \left[ \left( 1 - \frac{3r^2}{2R_2^2} \right) + \frac{3z(z+h)}{R_2^2} \left( 1 - \frac{5}{2} \frac{r^2}{R_2^2} \right) \right], \\ \sigma'_{rz} &= -\frac{12\mu A_0 r z}{R_2^5} \left( 1 - \frac{5(z+h)^2}{R_2^2} \right),\end{aligned}\quad (12)$$

$$\begin{aligned}\sigma_{rr}^1 &= \frac{4\mu A_0}{R_2^3} \left\{ \frac{2\lambda + \mu}{\lambda + \mu} - \frac{3z(z+h)}{R_2^2} - 3\frac{r^2}{R_2^2} + \frac{15r^2(z+h)z}{R_2^4} \right\}, \\ \sigma_{\theta\theta}^1 &= \frac{4\mu A_0}{R_2^3} \left\{ \frac{2\lambda + \mu}{\lambda + \mu} - \frac{3z(z+h)}{R_2^2} - \frac{3\lambda}{\lambda + \mu} \frac{r^2}{R_2^2} \right\};\end{aligned}\quad (12)$$

the integrals that arise from the Bessel functions have been calculated with the aid of the formulas given in (26, 3).

The sum of (4) and (5) with (12) also solves our problem:

$$\begin{aligned}u_z &= A_0 \left[ \frac{z-h}{R_1^3} - \left( \frac{z+h}{R_2^3} \frac{\lambda+3\mu}{\lambda+\mu} + \frac{4z}{R_2^3} - \frac{6zr^2}{R_2^5} \right) \right], \\ u_r &= A_0 r \left[ \frac{1}{R_1^3} + \frac{\lambda+3\mu}{\lambda+\mu} \frac{1}{R_2^3} - \frac{6z(z+h)}{R_2^5} \right], \\ \sigma_{zz} &= 4\mu A_0 \left\{ -\frac{1}{R_1^3} \left( 1 - \frac{3}{2} \frac{r^2}{R_1^2} \right) + \frac{1}{R_2^3} \left( 1 - \frac{3}{2} \frac{r^2}{R_2^2} \right) + \frac{6z(z+h)}{R_2^5} \left( 1 - \frac{5}{2} \frac{r^2}{R_2^2} \right) \right\}, \\ \sigma_{rr} &= 2\mu A_0 \left\{ \frac{1}{R_1^3} \left( 1 - 3\frac{r^2}{R_1^2} \right) + \frac{1}{R_2^3} \left( \frac{5\lambda+3\mu}{\lambda+\mu} - 9\frac{r^2}{R_2^2} \right) - \frac{6z(z+h)}{R_2^5} \left( 1 - 5\frac{r^2}{R_2^2} \right) \right\}, \\ \sigma_{\theta\theta} &= 2\mu A_0 \left\{ \frac{1}{R_1^3} + \frac{1}{R_2^3} \left( \frac{5\lambda+3\mu}{\lambda+\mu} - \frac{6\lambda}{\lambda+\mu} \frac{r^2}{R_2^2} \right) - \frac{6z(z+h)}{R_2^5} \right\}, \\ \sigma_{rz} &= -6\mu A_0 \left\{ \frac{r(z-h)}{R_1^5} + \frac{r(z+h)}{R_2^5} + \frac{2rz}{R_2^5} \left( 1 - \frac{5(z+h)^2}{R_2^2} \right) \right\}.\end{aligned}\quad (13)$$

4. The results can be applied to certain concrete physicochemical problems: the question of deep chemical adsorption (4), the formation of cracks near a surface, etc. In particular, one can calculate the insertion energy of an absolutely rigid sphere of radius  $R_0$  as a function of the distance  $h$  of its center from the surface. Modeling the insertion as a uniform expansion of the sphere ( $R_0 \rightarrow (1+\delta)R_0$ ), we obtain for the insertion work the expression

$$W = W_0 \frac{1-2q^3}{\left[ 1 - \frac{2q^3}{(1+q^2)^{3/2}} + \frac{3q^5}{(1+q^2)^{5/2}} \right]} \simeq W_0(1-6q^5 + o(q^6));\quad (14)$$

here

$$W_0 = 6\mu \left( \frac{4\pi R_0^3}{3} \right) \delta^2, \quad q = \frac{R_0}{2h}, \quad \lim_{x \rightarrow 0} x o(x) = 0.$$

Expression (14) makes it possible to conclude that  $W$  is a monotonic function of  $q$ , rapidly approaching  $W_0$  as  $q \rightarrow 0$  ( $W(1/2) \simeq 0.85W_0$ ,  $W(1/3) \simeq 0.98W_0$ ,  $W(1/4) \simeq 0.994W_0$ ).

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Physicochemical Institute  
named after L. Ya. Karpov

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*Note: Figure translations are in progress. See original paper for figures.*

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