NORMALLY SOLVABLE ELLIPTIC BOUNDARY-VALUE PROBLEMS

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Abstract

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NORMALLY SOLVABLE ELLIPTIC BOUNDARY-VALUE PROBLEMS

In a domain D of Euclidean space let us consider a linear elliptic differential operator Lu. Let Bu be a linear operator prescribed on the boundary S of the domain D.

The study of a broad class of boundary-value problems

$$Lu = h, \tag{1}$$

$$Bu = f \tag{2}$$

is reduced to an equivalent equation of the form

$$T\varphi = g, \tag{3}$$

where T is a linear operator mapping the linear metric space E_φ into the linear metric space E_g.

For certain classes of homogeneous boundary conditions

$$Bu = 0 \tag{4}$$

it is possible, in a natural way, to introduce the so-called homogeneous adjoint boundary-value problem, which plays an important role in the theory of the boundary-value problem (1)–(4).

In all these cases one rather easily obtains necessary solvability conditions for the problem under consideration in terms of the orthogonality of g (or h) to the space E_ψ of solutions of the homogeneous equation adjoint to (3) (or to the space E_v of solutions of the homogeneous adjoint boundary-value problem corresponding to (1)–(4)). When these conditions are not only necessary but also sufficient for the solvability of the problem (1)–(2) (or the problem (1)–

(4)), the problem under consideration will be called normally solvable in the sense of Hausdorff.

Let l denote the dimension of the space of solutions of the homogeneous equation $T\varphi = 0$ (or of the space of solutions of the homogeneous boundary-value problem $Lu = 0,\ Bu = 0$), and let l' denote the dimension of the space E_ψ (or of the space E_v). A problem normally solvable in the sense of Hausdorff will be called Noetherian if l and l' are both finite. A Noetherian problem will be called Fredholm if $l = l'$. The difference $l - l' = \chi$ is sometimes called the index of the Noetherian problem.

In our view, one of the central questions in the theory of elliptic boundary-value problems is the determination of criteria for normal solvability. No less important is the question of establishing the degree of overdetermination or the degree of underdetermination of the boundary-value problem under consideration. In resolving the latter question, the computation of the index χ of a Noetherian problem is undoubtedly important, but far from decisive.

Criteria for Noetherianity and Fredholmness have at present been established only for certain classes of elliptic boundary-value problems.

For a broad class of boundary-value problems, Noetherness is violated even in the case of elliptic operators with two independent variables. A simple example of problems that are normally solvable in the sense of Hausdorff but are not Noetherian is the Dirichlet problem

$$u_1(t) = u_2(t) = 0, \qquad |t| = 1, \tag{5}$$

for the elliptic system

$$\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial y^2} - 2\frac{\partial^2 u_2}{\partial x\,\partial y} = h_1,$$

$$2\frac{\partial^2 u_1}{\partial x\,\partial y} + \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial y^2} = h_2, \tag{6}$$

where h_1 and h_2 are continuously differentiable functions in the disk $\overline{D} : |z| \leq 1$.

The homogeneous adjoint boundary-value problem corresponding to (6)–(5) is defined as follows:

$$\frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial y^2} + 2\frac{\partial^2 v_2}{\partial x\,\partial y} = 0, \qquad -2\frac{\partial^2 v_1}{\partial x\,\partial y} + \frac{\partial^2 v_2}{\partial x^2} - \frac{\partial^2 v_2}{\partial y^2} = 0. \tag{7}$$

$$v_1(t) = v_2(t) = 0, \qquad |t| = 1; \tag{8}$$

All solutions of problem (7), (8) that are regular in the disk D and continuous in $\overline{D}$ are given by the formula

$$v_1 + iv_2 = (1 - z\bar{z})\, \psi_0(\bar{z}),$$

where $\psi_0(\bar{z})$ is an arbitrary holomorphic function in D, continuous in $\overline{D}$.

Let us now note that the general solution of system (6) is represented by the formula

$$u_1 + iu_2 = \bar{z}\,\varphi(z) + \psi(z) + w(z), \tag{9}$$

where $\varphi(z)$ and $\psi(z)$ are arbitrary holomorphic functions in D,

$$w(z) = \frac{1}{\pi^2} \iint_D \frac{d\xi\, d\eta}{t - z} \iint_D \frac{h(t_1)\, d\xi_1\, d\eta_1}{t_1 - t}, \qquad t = \xi + i\eta, \quad t_1 = \xi_1 + i\eta_1,$$

$$h = h_1 + ih_2.$$

On the basis of (5) and (9) we conclude that problem (6)–(5) is solvable if and only if $tw(t)$ is the boundary value of a function holomorphic in the disk D as $z \to t$. This condition, in turn, is equivalent to the conditions

$$\int_{|t|=1} t^k w(t)\, dt = 0, \qquad k = 1, 2, \ldots \tag{10}$$

When conditions (10) are satisfied, the solution of problem (6)–(5) is written explicitly as

$$u_1 + iu_2 = (1 - z\bar{z})\varphi(z) + w(z) - \frac{\bar{z}}{2\pi i} \int_{|t|=1} \frac{tw(t)\, dt}{t - z}.$$

In view of the fact that

$$\frac{1}{\pi} \iint_D \frac{t^k\, d\xi\, d\eta}{t - z} = z^{k-1}(1 - z\bar{z}), \qquad k = 1, 2, \ldots,$$

conditions (10) take the form

$$\iint_D h(t)(1 - \bar{t}t)t^k\, d\xi\, d\eta = 0, \qquad k = 0, 1, 2, \ldots \tag{11}$$

Separating the real and imaginary parts of equalities (11), we immediately verify that equalities (10) are equivalent to the conditions of orthogonality over the disk D of the right-hand sides of (6) to all solutions of the adjoint homogeneous problem (7), (8). That is, problem (6)–(5), while not Noetherian, is normally solvable in the sense of Hausdorff (it is Hausdorff).

Let now D be a simply connected finite domain whose boundary S contains the segment $a < x < b$ of the real axis. The Dirichlet problem

$$u_1(t) = u_2(t) = 0, \quad t \in S, \tag{12}$$

for system (6) ceases to be normally solvable.

Indeed, the homogeneous adjoint problem corresponding to (6)–(12) has only the trivial solution $v_1 \equiv v_2 \equiv 0$. Therefore the necessary condition for solvability of problem (6)–(12)

$$\iint_D (v_1 h_1 + v_2 h_2)\, dx\, dy = 0 \tag{13}$$

is satisfied automatically. On the other hand, by virtue of (12) and (9) we must have

$$x\varphi(x) + \psi(x) + w(x) = 0, \quad a < x < b.$$

But this is impossible for arbitrary h_1 and h_2 continuously differentiable in $D + S$.

Consequently, problem (6)–(12) in the case under consideration is not unconditionally solvable, and, therefore, satisfaction of condition (13) cannot ensure its solvability.

The identification of classes of elliptic systems and domains D for which the Dirichlet problem is normally solvable in the sense of Hausdorff is undoubtedly of scientific interest.

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Note: Figure translations are in progress. See original paper for figures.