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Abstract

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A CONDITION FOR THE PRESERVATION OF METRIZABILITY UNDER QUOTIENT MAPPINGS

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The question of when, under mappings, one or another property of topological spaces is preserved is one of the most familiar problems of general topology. Among the most interesting properties of general spaces there belongs, without doubt, the property of metrizability; metrizability, in particular, always means that the space has a number of important special topological properties (paracompactness, perfect normality, etc.). The determination of conditions for the preservation of metrizability under mappings is apparently a difficult problem. It is possible that in the class of all continuous mappings this problem has no sufficiently meaningful solution. A strong special result is due to A. H. Stone: he proved that metrizability is preserved under closed bicompact mappings * (5). Below a criterion will be given for the metrizability of an arbitrary quotient space of a metrizable space. Stone' s theorem is obtained as a special case of this general result.

The principal role in what follows is played by the notion of a regular mapping ** (see (2)).

Regular mappings are extremely widespread—it can be shown that every space is a regular image of some metric space. The breadth of this notion is also indicated by the following

Proposition 1. *Every continuous mapping of a metrizable space X onto a metrizable Y is regular with respect to some metric on X.*

The latter circumstance gives grounds for expecting that the notion of a regular mapping might find its place in the formulation of a criterion for the preservation of metrizability under mappings. This idea is fully confirmed by the following main

Theorem. *The quotient space of a metrizable space is metrizable if and only if the corresponding mapping is regular.*

Proof of this theorem is the sole aim of the present paper ***. We begin with the proof of Proposition 1.

Lemma 1. *Let $\varphi = \{\gamma_n\}$ be a refining sequence of covers of a metrizable space X. Then there exists on X a metric ρ such that if $\rho(x, y) < 1/2^n$, where $x, y \in X$, $n > 0$ is an integer, then there is a $U \in \gamma_n$ for which $x \in U$, $y \in U$. (Such a metric will be called a metric associated with the system φ.)*

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* However, even if only one element of a partition of the plane is nontrivial, the space of the partition may fail to be metrizable.

** A mapping $f : X \to Y$ of a metrizable space X onto a topological space Y is called regular with respect to a metric ρ, given on X, if for every point $y \in Y$ and every neighborhood O_y of it there is a neighborhood $O_1 y$ such that $\rho(f^{-1}O_1 y,\ X \setminus f^{-1}O_y) > 0$. See in this connection the definition of a weakly uniformly regular mapping given by V. I. Ponomarev (3).

*** As will be seen, an essential role in the proof is played by the criterion of metrizability proved in (1) and based on the notion of a fundamental system of covers, which we shall regard as known (see (1)).

Proof. Without loss of generality, we may assume that all covers γ_n are locally finite. This is proved by applying Stone's theorem. For each cover $\gamma_n \equiv \varphi$, $\gamma_n = \{U^n_\alpha \mid \alpha \in M_n\}$, choose $\gamma'_n = \{V^n_\alpha \mid \alpha \in M_n\}$—some open cover of the space X such that $[V^n_\alpha] \subseteq U^n_\alpha$, $\alpha \in M_n$. Such a γ'_n, as is well known, exists (see (4)). Let $f_{n,\alpha}$ be some function continuous on X such that $f_{n,\alpha}(x) = 1$ for $x \in [V^n_\alpha]$ and $f_{n,\alpha}(x) = 0$ for $x \in X \setminus U^n_\alpha$. Such a function exists, since the space X is normal, and $[V^n_\alpha]$ and $X \setminus U^n_\alpha$ are disjoint closed subsets of it.

Put, for arbitrary points $x, y \in X$,

$$d_n(x, y) = \min\left(\sum_{\alpha \in M_n} \left|f_{n,\alpha}(x) - f_{n,\alpha}(y)\right|,\ 1/2^n\right).$$

Obviously, $d_n(x, y)$ is a pseudometric on X. The function $d_n(x, y)$ depends continuously on x and y. Put

$$\rho(x, y) = \sum_{n=1}^{\infty} d_n(x, y).$$

The series

$$\sum_{n=1}^{\infty} d_n(x, y)$$

converges uniformly on $X \times X$ (for all its terms are positive, and it is majorized by the series $\sum 1/2^n$). Therefore $\rho(x, y)$ is everywhere a finite function continuous on $X \times X$. Obviously, $\rho(x, y)$ is a pseudometric on X (since all $d_n(x, y)$ are pseudometrics on X).

We show that $\rho(x, y)$ is in fact a metric on the space X, agreeing with its topology. This means, first, that if a point $x_0 \in X$ and its neighborhood Ox_0

are given arbitrarily, then we must be able to indicate such a natural number n that $O_{1/2^n}(x_0) \subseteq Ox_0$. We do this. We can choose a number N so that $\gamma_N(x_0) \subseteq Ox_0$ (since the sequence $\{\gamma_n\}$ is refining). For some $a_0 \in M_N$ we have $V^N_{a_0} \ni x_0$, $V^N_{a_0} \in \gamma'_N$. Then $Ox_0 \supseteq U^N_{a_0} \supseteq [V^N_{a_0}] \ni x_0$. But if $y \in X \setminus Ox_0$, then

$$|f_{N,a_0}(x_0) - f_{N,a_0}(y)| = 1.$$

Therefore

$$\sum_{\alpha \in M_N} |f_{N,\alpha}(x_0) - f_{N,\alpha}(y)| \geq 1$$

and, consequently,

$$d_N(x_0, y) = 1/2^N \quad \text{and} \quad \rho(x_0, y) \geq 1/2^N.$$

Since the point $y \in X \setminus Ox_0$ was chosen arbitrarily, the last relation means precisely that

$$O_{1/2^N}(x_0) \subseteq Ox_0.$$

Now let $x \neq y$; $x, y \in X$. There exists (since X is a T_1-space) a neighborhood $Ox \ni x$, $Ox \not\ni y$. We have shown that, for some N,

$$O_{1/2^N}(x) \subseteq Ox.$$

But then, since $y \in X \setminus Ox$, we have $\rho(x, y) \geq 1/2^N > 0$. Thus it has been proved that the pseudometric ρ is a metric on X.

It remains for us to verify that for an arbitrary natural number n there exists a neighborhood Ox of the point x such that

$$x \in Ox \subseteq O_{1/n,x}.$$

But this follows directly from the fact that the function $\rho(x, y)$ is continuous on $X \times X$.

We have shown that the metric ρ agrees with the topology of the space X. Let us now check whether it has the special property we need. Suppose that the points $x, y \in X$ are chosen so that, in some cover $\gamma_N \equiv \varphi$, there is no element containing x and y simultaneously. We show that then $\rho(x, y) \geq 1/2^N$. Indeed, there is an element $V^N_{a_0} \ni x$, $V^N_{a_0} \in \gamma'_N$. By the earlier supposition $y \in X \setminus U^N_{a_0}$. Then

$$|f_{N,a_0}(x) - f_{N,a_0}(y)| = 1.$$

Consequently,

$$d_N(x, y) = 1/2^N \quad \text{and} \quad \rho(x, y) \geq 1/2^N.$$

Lemma 1 is proved.

Let us return to the proof of Proposition 1. It is easy to show that in the spaces X and Y there exist such fundamental sets of covers $\{\gamma_n\}$ and $\{\lambda_n\}$ that $f\gamma_n < \lambda_n$ for each n. Define on X some metric associated with the system $\{\gamma_n\}$.

Denote it by ρ. We show that the mapping f is regular with respect to this metric. Indeed, let arbitrary points $y_0 \in Y$ and its neighborhood Oy_0 be taken. Consider the sets $O\Phi_0 = f^{-1}Oy_0$ and $\Phi_0 = f^{-1}y_0$. There exists a neighborhood O_1y_0 for which, for some natural number n_0,

$$\lambda_{n_0}(O_1y_0) \subseteq Oy_0.$$

Then, if $U \in \gamma_{n_0}$ and

$U \cap f^{-1}O_1y_0 \neq \Lambda$, then $U \subseteq O\Phi_0$. Indeed, $fU \cap O_1y_0 \neq \Lambda$. But $fU \subseteq V$ for some $V \in \lambda_{n_0}$, and therefore $fU \subseteq Oy_0$ by the choice of the neighborhood O_1y_0. Hence, $U \subseteq f^{-1}Oy_0 = O\Phi_0$, or $U \cap (X \setminus f^{-1}Oy_0) = \Lambda$. We shall show that

$$\rho(X \setminus f^{-1}Oy_0,\ f^{-1}O_1y_0) \geq 1/2^{n_0} > 0,$$

which will complete the proof of Proposition 1. But this is so, for if $x_1 \in X$ and $x_2 \in X$ are any points such that $x_1 \in f^{-1}O_1y_0$ and $x_2 \in X \setminus f^{-1}Oy_0$, then, by what has been proved, from $U \in \gamma_{n_0}$ it follows that at least either $U \not\ni x_1$ or $U \not\ni x_2$. Then, by the definition of the metric ρ,

$$\rho(x_1, x_2) \geq 1/2^{n_0}.$$

But in that case

$$\rho(X \setminus f^{-1}Oy_0,\ f^{-1}O_1y_0) = \inf_{x_1 \in f^{-1}O_1y_0,\ x_2 \in X \setminus f^{-1}Oy_0} \rho(x_1, x_2) \geq 1/2^{n_0} > 0,$$

which also means that the mapping f is regular (with respect to the metric ρ). Proposition 1 is completely proved.

Proposition 2. *Let the mapping $f : X \to Y$ of a metric space X onto a T_0-space Y satisfy two conditions: 1) f is regular; 2)* $\operatorname{Int} fO_\varepsilon f^{-1}y \ni y$*, whatever the point $y \in Y$ and the number $\varepsilon > 0$. Then Y is metrizable.*

Proof. Consider $\varphi = \{\gamma_n\}$, where γ_n, $n = 1, 2, \ldots$, consists of all open subsets of X whose diameter does not exceed $1/n$. For each n, denote by λ_n the system of subsets of the space Y consisting of the kernels of the stars of the points of this space with respect to the collection of images of elements of the cover γ_n:

$$\lambda_n = \{\operatorname{Int} f\gamma_n(y) \mid y \in Y\}.$$

We shall show that λ_n, $n = 1, 2, \ldots$, together form a fundamental set of covers of the space Y.

First of all, the λ_n are covers of Y:

$$\operatorname{Int} f\gamma_n(y) = \operatorname{Int} f(\gamma_n(f^{-1}y)) \ni y$$

by condition 2). Let now y_0 and Oy_0 be an arbitrary point and its neighborhood in the space Y. Since the mapping f is regular, there will be neighborhoods O_1y_0 and O_2y_0 of the point y_0 such that

$$\rho(f^{-1}O_1y_0,\ X \setminus f^{-1}Oy_0) > 0, \qquad \rho(f^{-1}O_2y_0,\ X \setminus f^{-1}O_1y_0) > 0.$$

Denote by r the smaller of the numbers appearing on the left-hand sides of the last inequalities, and choose N so that $1/N < r$. We shall now prove that

$$\lambda_N(O_2y_0) \subseteq Oy_0.$$

Indeed, let $y \in \lambda_N(O_2y_0)$. This, as follows obviously from the definition of the system λ_N, means that in γ_N there are such elements $G^N_{\alpha_1}$ and $G^N_{\alpha_2}$ that the following relations are simultaneously satisfied: $1'$) $fG^N_{\alpha_2} \cap O_2y_0 \neq \Lambda$, $2'$) $fG^N_{\alpha_1} \cap fG^N_{\alpha_2} \neq \Lambda$, and $3'$) $fG^N_{\alpha_1} \ni y$. Then from $1'$) $G^N_{\alpha_2} \cap f^{-1}O_2y_0 \neq \Lambda$, and consequently, since

$$\operatorname{diam} G^N_{\alpha_2} \leq 1/N < r,$$

$$G^N_{\alpha_2} \subseteq f^{-1}O_1y_0.$$

Hence $fG^N_{\alpha_2} \subseteq O_1y_0$, and therefore

$$(fG^N_{\alpha_1} \cap fG^N_{\alpha_2}) \subseteq O_1y_0.$$

Since the set standing in parentheses in the last relation is nonempty (see $2'$)), we may write

$$fG^N_{\alpha_1} \cap O_1y_0 \neq \Lambda$$

or, equivalently,

$$G^N_{\alpha_1} \cap f^{-1}O_1y_0 \neq \Lambda.$$

Since

$$\operatorname{diam} G^N_{\alpha_1} \leq 1/N < r,$$

it follows that

$$G^N_{\alpha_1} \subseteq f^{-1}Oy_0,$$

and hence $fG^N_{\alpha_1} \subseteq Oy_0$. Consequently, $y \in Oy_0$—by $3'$). Since y is an arbitrary point of the set $\lambda_N(O_2y_0)$, this proves that $\lambda_N(O_2y_0) \subseteq Oy_0$. This means that the system $\{\lambda_n\}$ forms a fundamental set of covers. Consequently, the space Y is metrizable by the theorem from (1). Proposition 2 is proved.

Lemma 2. *Let $f : X \to Y$ be a quotient regular mapping of a metric space X onto a Hausdorff space Y. Then, if the set $M \subseteq X$ satisfies the condition $f^{-1}fM = M$, then $f^{-1}f[M]$ is closed.*

Proof. Suppose the contrary, and let

$$x_0 \in f^{-1}f[M] \setminus f^{-1}[M].$$

Put

$$L = fM, \qquad y_0 = fx_0, \qquad \Phi_0 = f^{-1}x_0.$$

Then

$$y_0 \in [f(f^{-1}f[M])] = [f[M]] = [L], \qquad \Phi_0 \cap f^{-1}f[M] = \Lambda.$$

We shall say that a point $x \in M$ is ε-attainable from x_0 if, for some $y \in Y$,

$$O_\varepsilon(f^{-1}y) \supseteq (x_0 \cup x)^*.$$

We shall show that for every $\varepsilon > 0$ there is in M a point ε-attainable from x_0. Indeed, it is enough to take $x_1 \in f^{-1}f[M]$ for which

$$\rho(x_0, x_1) < \varepsilon;$$

let $y_1 = fx_1$. Then $f^{-1}y_1 \cap [M] \neq \Lambda$, i.e. there exists

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* By $O_\varepsilon(A)$, where A is a set, is denoted the set of points whose distance from A is less than ε.

$x_2 \in [M]$, for which $fx_2 = fx_1 = y_1$. In M one can find a point x_3 such that $\rho(x_3, x_2) < \varepsilon$. We now have $O_\varepsilon(f^{-1}y_1) \supset (x_1 \cup x_3)$, which proves that the point x_3 is ε-reachable from x_0.

Let us continue the proof of the lemma. For each n choose in M some point x_n that is $1/n$-reachable from x_0. We assert that then

$$\left[\bigcup_{n=1}^{\infty} x_n\right] \cap \Phi_0 \neq \Lambda.$$

First let us establish that $\{fx_n\}$ converges to y_0. Consider an arbitrary neighborhood Oy_0 of the point y_0. Choose O_1y_0 so that $\rho(f^{-1}O_1y_0, X \setminus f^{-1}Oy_0) > 0$. Then also $\rho(f^{-1}y_0, X \setminus f^{-1}O_1y_0) > 0$.

Let now the number N satisfy the condition

$$\frac{1}{N} < \min\left\{\rho(f^{-1}O_1y_0, X \setminus f^{-1}Oy_0),\ \rho(f^{-1}y_0, X \setminus f^{-1}O_1y_0)\right\}.$$

It turns out that $fx_n \in Oy_0$ for $n > N$. Indeed, the point y_n for which $O_{1/n}(f^{-1}y_n) \supset (x_0 \cup x_n)$ then belongs to O_1y_0—this follows from the relations $1/n < 1/N < \rho(f^{-1}y_0, X \setminus f^{-1}O_1y_0)$. Hence $f^{-1}y_n \subset f^{-1}O_1y_0$. From the inequality $1/n < \rho(f^{-1}O_1y_0, X \setminus f^{-1}Oy_0)$, in view of the fact that $O_1(f^{-1}O_1y_0) \supset x_n$, we obtain that $x_n \in f^{-1}Oy_0$, whence $fx_n \in Oy_0$. Thus it is proved that the sequence $\{fx_n\}$ converges to y_0. Since Y is a Hausdorff space, y_0 is the unique limit point of the set $\{fx_n\}$; therefore the set P, consisting of the points of the sequence $\{fx_n\}$ and the point y_0, is closed in Y. Then the set $Q = f^{-1}P \cap [M]$ is closed in X. Obviously, $f^{-1}P = Q \cup \Phi_0$, and if we assume that $\Phi_0 \cap [M]$ is empty, then we obtain that

$$Q = f^{-1}\left(\bigcup_{n=1}^{\infty} fx_n\right),$$

where Q is closed in X, while $\bigcup_{n=1}^{\infty} fx_n$ is a nonclosed set in Y. This contradicts the factorness of f. Lemma 2 is completely proved.

Let us derive from Lemma 2 that every regular quotient mapping of a metric space satisfies condition 2 of Proposition 2*. Indeed, let y be any point of Y, and let U be some open set in X containing $f^{-1}y$. Put $L = Y \setminus fU$ and $M = f^{-1}L$. Then $f^{-1}fM = M$ and $U \cap M = \Lambda$. Hence $f^{-1}y \cap [M] = \Lambda$, i.e. $y \notin f[M]$. Applying Lemma 2 to the set M, we obtain that $f^{-1}f[M]$ is closed in X; hence, by the factorness of f, $f[M]$ is closed in Y. We have $y \in Y \setminus f[M] \subseteq Y \setminus fM \subseteq Y \setminus L \subseteq fU$, which proves that $\operatorname{Int} fU \ni y$. The theorem now follows from Proposition 2.

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* At first this assertion was derived from a much more intricate chain of lemmas formulated by me, the author of the proof of one of which was M. Čoban. Here they have been rationally merged.

Note: Figure translations are in progress. See original paper for figures.